

FOURIER TRANSFORM OF SUPERPOSITION OF PLANE WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.32.

The general solution to the wave equation can be written as a superposition of plane waves:

$$\tilde{f}(z, t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz - \omega t)} dk \quad (1)$$

where $\tilde{A}(k)$ is the (complex) amplitude of waves with wave number k , that is, it's the contribution to the overall wave \tilde{f} of waves with a given wave number. At $t = 0$, this is

$$\tilde{f}(z, 0) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{ikz} dk \quad (2)$$

and its time derivative at $t = 0$ is

$$\dot{\tilde{f}}(z, 0) = -i \int_{-\infty}^{\infty} \omega(k) \tilde{A}(k) e^{ikz} dk \quad (3)$$

Note that we can't take ω outside the integral as it is, in general, a function of k : $\omega = \omega(k)$.

According to Plancherel's theorem, the Fourier transform and its inverse of a function $\phi(k)$ are

$$\tilde{\phi}(z) = \int_{-\infty}^{\infty} \tilde{\Phi}(k) e^{ikz} dk \quad (4)$$

$$\tilde{\Phi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(z) e^{-ikz} dz \quad (5)$$

Therefore, the inverse transforms of 2 and 3 are

$$\tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(z, 0) e^{-ikz} dz \quad (6)$$

$$-i\omega(k)\tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \dot{\tilde{f}}(z, 0) e^{-ikz} dz \quad (7)$$

$$\int_{-\infty}^{\infty} \tilde{f}(z, 0) e^{-ikz} dz = \int_{-\infty}^{\infty} \frac{i}{\omega} \dot{\tilde{f}}(z, 0) e^{-ikz} dz \quad (8)$$

In this case, we *can* put ω inside the integral in the last line, since it doesn't depend on z .

Now let

$$\tilde{f} = f + ig \quad (9)$$

where f and g are real functions. Then from 8, the real and imaginary parts must be equal on each side, so

$$\int_{-\infty}^{\infty} (f + ig) e^{-ikz} dz = \int_{-\infty}^{\infty} \frac{i}{\omega} (\dot{f} + i\dot{g}) e^{-ikz} dz \quad (10)$$

$$= \int_{-\infty}^{\infty} \left(\frac{i}{\omega} \dot{f} - \frac{1}{\omega} \dot{g} \right) e^{-ikz} dz \quad (11)$$

$$f = -\frac{\dot{g}}{\omega} \quad (12)$$

$$g = \frac{\dot{f}}{\omega} \quad (13)$$

[We can ignore the e^{-ikz} in these calculations since the same factor appears in both integrals, so in order for the real and imaginary parts of the two integrands to be equal, the factor multiplying e^{-ikz} must have equal real and imaginary parts on both sides.]

Substituting this back into 6 we get

$$\tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f(z, 0) + \frac{i}{\omega} \dot{f}(z, 0) \right] e^{-ikz} dz \quad (14)$$