

## FOURIER TRANSFORM OF SUPERPOSITION OF PLANE WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.32.

The general solution to the wave equation can be written as a superposition of plane waves:

$$(1) \quad \tilde{f}(z, t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz - \omega t)} dk$$

where  $\tilde{A}(k)$  is the (complex) amplitude of waves with wave number  $k$ , that is, it's the contribution to the overall wave  $\tilde{f}$  of waves with a given wave number. At  $t = 0$ , this is

$$(2) \quad \tilde{f}(z, 0) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{ikz} dk$$

and its time derivative at  $t = 0$  is

$$(3) \quad \dot{\tilde{f}}(z, 0) = -i \int_{-\infty}^{\infty} \omega(k) \tilde{A}(k) e^{ikz} dk$$

Note that we can't take  $\omega$  outside the integral as it is, in general, a function of  $k$ :  $\omega = \omega(k)$ .

According to Plancherel's theorem, the Fourier transform and its inverse of a function  $\phi(k)$  are

$$(4) \quad \tilde{\phi}(z) = \int_{-\infty}^{\infty} \tilde{\Phi}(k) e^{ikz} dk$$

$$(5) \quad \tilde{\Phi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(z) e^{-ikz} dz$$

Therefore, the inverse transforms of 2 and 3 are

$$(6) \quad \tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(z, 0) e^{-ikz} dz$$

$$(7) \quad -i\omega(k)\tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \dot{\tilde{f}}(z, 0) e^{-ikz} dz$$

$$(8) \quad \int_{-\infty}^{\infty} \tilde{f}(z, 0) e^{-ikz} dz = \int_{-\infty}^{\infty} \frac{i}{\omega} \dot{\tilde{f}}(z, 0) e^{-ikz} dz$$

In this case, we *can* put  $\omega$  inside the integral in the last line, since it doesn't depend on  $z$ .

Now let

$$(9) \quad \tilde{f} = f + ig$$

where  $f$  and  $g$  are real functions. Then from 8, the real and imaginary parts must be equal on each side, so

$$(10) \quad \int_{-\infty}^{\infty} (f + ig) e^{-ikz} dz = \int_{-\infty}^{\infty} \frac{i}{\omega} (\dot{f} + i\dot{g}) e^{-ikz} dz$$

$$(11) \quad = \int_{-\infty}^{\infty} \left( \frac{i}{\omega} \dot{f} - \frac{1}{\omega} \dot{g} \right) e^{-ikz} dz$$

$$(12) \quad f = -\frac{\dot{g}}{\omega}$$

$$(13) \quad g = \frac{\dot{f}}{\omega}$$

[We can ignore the  $e^{-ikz}$  in these calculations since the same factor appears in both integrals, so in order for the real and imaginary parts of the two integrands to be equal, the factor multiplying  $e^{-ikz}$  must have equal real and imaginary parts on both sides.]

Substituting this back into 6 we get

$$(14) \quad \tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ f(z, 0) + \frac{i}{\omega} \dot{f}(z, 0) \right] e^{-ikz} dz$$