FOURIER TRANSFORM OF SUPERPOSITION OF PLANE WAVES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.32.

The general solution to the wave equation can be written as a superposition of plane waves:

$$\tilde{f}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)\,e^{i(kz-\omega t)}dk \tag{1}$$

where $\tilde{A}(k)$ is the (complex) amplitude of waves with wave number k, that is, it's the contribution to the overall wave $\tilde{f}$ of waves with a given wave number. At $t=0$, this is

$$\tilde{f}(z,0)=\int_{-\infty}^{\infty}\tilde{A}(k)\,e^{ikz}dk \tag{2}$$

and its time derivative at $t=0$ is

$$\dot{\tilde{f}}(z,0)=-i\int_{-\infty}^{\infty}\omega(k)\tilde{A}(k)\,e^{ikz}dk \tag{3}$$

Note that we can't take ω outside the integral as it is, in general, a function of k: $\omega=\omega(k)$.

According to Plancherel's theorem, the Fourier transform and its inverse of a function $\phi(k)$ are

$$\begin{aligned}\tilde{\phi}(z) &= \int_{-\infty}^{\infty}\tilde{\Phi}(k)\,e^{ikz}dk && (4)\\ \tilde{\Phi}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\phi}(z)\,e^{-ikz}dz && (5)\end{aligned}$$

Therefore, the inverse transforms of 2 and 3 are

$$\begin{aligned}
\tilde{A}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(z,0)\,e^{-ikz}dz &(6)\\
-i\omega(k)\tilde{A}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\dot{\tilde{f}}(z,0)\,e^{-ikz}dz &(7)\\
\int_{-\infty}^{\infty}\tilde{f}(z,0)\,e^{-ikz}dz &= \int_{-\infty}^{\infty}\frac{i}{\omega}\dot{\tilde{f}}(z,0)\,e^{-ikz}dz &(8)
\end{aligned}$$

In this case, we *can* put ω inside the integral in the last line, since it doesn't depend on z.

Now let

$$\tilde{f} = f + ig \tag{9}$$

where f and g are real functions. Then from 8, the real and imaginary parts must be equal on each side, so

$$\begin{aligned}
\int_{-\infty}^{\infty}(f+ig)\,e^{-ikz}dz &= \int_{-\infty}^{\infty}\frac{i}{\omega}\left(\dot{f}+i\dot{g}\right)e^{-ikz}dz &(10)\\
&= \int_{-\infty}^{\infty}\left(\frac{i}{\omega}\dot{f}-\frac{1}{\omega}\dot{g}\right)e^{-ikz}dz &(11)\\
f &= -\frac{\dot{g}}{\omega} &(12)\\
g &= \frac{\dot{f}}{\omega} &(13)
\end{aligned}$$

[We can ignore the e^{-ikz} in these calculations since the same factor appears in both integrals, so in order for the real and imaginary parts of the two integrands to be equal, the factor multiplying e^{-ikz} must have equal real and imaginary parts on both sides.]

Substituting this back into 6 we get

$$\tilde{A}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(z,0)+\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}dz \tag{14}$$