

SPHERICAL ELECTROMAGNETIC WAVE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 9, Post 33.

One form of spherical wave has an electric component given by

$$(1) \quad \mathbf{E}(r, \theta, \phi, t) = A \frac{\sin \theta}{r} \left[\cos(kr - \omega t) - \frac{\sin(kr - \omega t)}{kr} \right] \hat{\phi}$$

where A is a constant. We can derive the corresponding magnetic field from Maxwell's equations in vacuum:

$$(2) \quad \nabla \cdot \mathbf{E} = 0$$

$$(3) \quad \nabla \cdot \mathbf{B} = 0$$

$$(4) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$(5) \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The calculations are straightforward but can get messy, so we'll use Maple to do the derivatives. We get

$$(6) \quad \nabla \times \mathbf{E} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\phi) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \hat{\boldsymbol{\theta}}$$

$$(7) \quad = \frac{2A \cos \theta}{r^2} \left[\cos(kr - \omega t) - \frac{\sin(kr - \omega t)}{kr} \right] \hat{\mathbf{r}} + \frac{A \sin \theta}{r} \left[\left(k - \frac{1}{kr^2} \right) \sin(kr - \omega t) + \frac{\cos(kr - \omega t)}{r} \right] \hat{\boldsymbol{\theta}}$$

Integrating this with respect to t we get

$$(8) \quad \mathbf{B} = \frac{2A \cos \theta}{r^2 \omega} \left[\sin(kr - \omega t) + \frac{\cos(kr - \omega t)}{kr} \right] \hat{\mathbf{r}} +$$

$$(9) \quad \frac{A \sin \theta}{r^3 \omega} \left[\left(\frac{1}{k} - kr^2 \right) \cos(kr - \omega t) + r \sin(kr - \omega t) \right] \hat{\boldsymbol{\theta}}$$

We can verify that \mathbf{E} and \mathbf{B} satisfy the other 3 Maxwell equations by direct calculation.

$$(10) \quad \nabla \cdot \mathbf{E} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (E_\phi) = 0$$

$$(11) \quad \nabla \cdot \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 B_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B_\theta)$$

$$(12) \quad = \frac{2A \cos \theta}{r^2 \omega} \left[\left(k - \frac{1}{kr^2} \right) \cos(kr - \omega t) - \frac{\sin(kr - \omega t)}{r} \right] +$$

$$(13) \quad \frac{2A \cos \theta}{r^2 \omega} \left[\frac{1}{r^2} \left(\frac{1}{k} - kr^2 \right) \cos(kr - \omega t) + \frac{\sin(kr - \omega t)}{r} \right]$$

$$(13) \quad = 0$$

$$(14) \quad \nabla \times \mathbf{B} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r B_\theta) - \frac{\partial B_r}{\partial \theta} \right] \hat{\phi}$$

$$(15) \quad = -\frac{2A \sin \theta}{\omega r^3} \left(\left(\frac{1}{kr} - kr \right) \cos(kr - \omega t) + \sin(kr - \omega t) \right) \hat{\phi} +$$

$$\frac{A \sin \theta}{\omega r^3} \left[-kr \cos(kr - \omega t) - \left(\frac{1}{k} - kr^2 - 1 \right) \sin(kr - \omega t) \right] \hat{\phi} +$$

$$\frac{2A \sin \theta}{\omega r^3} \left[\sin(kr - \omega t) + \frac{\cos(kr - \omega t)}{kr} \right] \hat{\phi}$$

$$(16) \quad = A \frac{\sin \theta}{r^2 c} [kr \sin(kr - \omega t) + \cos(kr - \omega t)] \hat{\phi}$$

where in the last line we used $\omega/k = c$ and collected terms. From 1 we get

$$(17) \quad \frac{\partial \mathbf{E}}{\partial t} = A \frac{\sin \theta}{r} \left[\omega \sin(kr - \omega t) + \frac{\omega \cos(kr - \omega t)}{kr} \right] \hat{\phi}$$

$$(18) \quad = A \frac{\sin \theta}{r^2} [krc \sin(kr - \omega t) + c \cos(kr - \omega t)] \hat{\phi}$$

$$(19) \quad = c^2 \nabla \times \mathbf{B}$$

so the final Maxwell equation is satisfied.

The Poynting vector is, after simplifying

(20)

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \\ &= \hat{\mathbf{r}} \frac{A^2 \sin^2 \theta}{\mu_0 \omega k r^5} \left[\left(\frac{1}{2} - r^2 k^2 \right) \sin [2(kr - \omega t)] - kr \cos [2(kr - \omega t)] + k^3 r^3 \cos^2 (kr - \omega t) \right] + \end{aligned}$$

(21)

$$\hat{\boldsymbol{\theta}} \frac{A^2 \sin 2\theta}{\mu_0 \omega k r^5} \left[\frac{1}{2} (k^2 r^2 - 1) \sin [2(kr - \omega t)] + kr \cos [2(kr - \omega t)] \right]$$

The intensity, or time average of the Poynting vector is

$$(22) \quad \mathbf{I} = \langle \mathbf{S} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathbf{S} dt$$

$$(23) \quad = \frac{A^2 k \sin^2 \theta}{2\mu_0 \omega r^2} \hat{\mathbf{r}}$$

The energy flows radially outwards and falls off as r^{-2} .

The total power radiated is the integral of $\mathbf{I} \cdot d\mathbf{a}$ over a sphere:

$$(24) \quad P = \int \mathbf{I} \cdot d\mathbf{a}$$

$$(25) \quad = \pi \frac{A^2 k}{\mu_0 \omega} \int_0^\pi \frac{r^2}{r^2} \sin^3 \theta d\theta$$

$$(26) \quad = \frac{4\pi A^2 k}{3\mu_0 \omega}$$

$$(27) \quad = \frac{4\pi A^2}{3\mu_0 c}$$

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