SPHERICAL ELECTROMAGNETIC WAVE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 9, Post 33.

One form of spherical wave has an electric component given by

$$\mathbf{E}(r,\theta\phi,t)=A\frac{\sin\theta}{r}\left[\cos(kr-\omega t)-\frac{\sin(kr-\omega t)}{kr}\right]\hat{\phi} \tag{0.1}$$

where A is a constant. We can derive the corresponding magnetic field from Maxwell's equations in vacuum:

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 0 &(0.2)\\
\nabla\cdot\mathbf{B} &= 0 &(0.3)\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} &(0.4)\\
\nabla\times\mathbf{B} &= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} &(0.5)
\end{aligned}$$

The calculations are straightforward but can get messy, so we'll use Maple to do the derivatives. We get

$$\begin{aligned}
\nabla\times\mathbf{E} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right)\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\theta} &(0.6)\\
&= \frac{2A\cos\theta}{r^2}\left[\cos(kr-\omega t)-\frac{\sin(kr-\omega t)}{kr}\right]\hat{\mathbf{r}}+ &(0.7)\\
&\quad \frac{A\sin\theta}{r}\left[\left(k-\frac{1}{kr^2}\right)\sin(kr-\omega t)+\frac{\cos(kr-\omega t)}{r}\right]\hat{\theta}
\end{aligned}$$

Integrating this with respect to t we get

$$\begin{aligned}
\mathbf{B} &= \frac{2A\cos\theta}{r^2\omega}\left[\sin(kr-\omega t)+\frac{\cos(kr-\omega t)}{kr}\right]\hat{\mathbf{r}}+ &(0.8)\\
&\quad \frac{A\sin\theta}{r^3\omega}\left[\left(\frac{1}{k}-kr^2\right)\cos(kr-\omega t)+r\sin(kr-\omega t)\right]\hat{\theta} &(0.9)
\end{aligned}$$

We can verify that **E** and **B** satisfy the other 3 Maxwell equations by direct calculation.

(0.10)
$$\nabla\cdot\mathbf{E}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right)=0$$

(0.11)
$$\nabla\cdot\mathbf{B}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta B_\theta\right)$$

(0.12)
$$=\frac{2A\cos\theta}{r^2\omega}\left[\left(k-\frac{1}{kr^2}\right)\cos\left(kr-\omega t\right)-\frac{\sin\left(kr-\omega t\right)}{r}\right]+\frac{2A\cos\theta}{r^2\omega}\left[\frac{1}{r^2}\left(\frac{1}{k}-kr^2\right)\cos\left(kr-\omega t\right)+\frac{\sin\left(kr-\omega t\right)}{r}\right]$$

(0.13)
$$=0$$

(0.14)
$$\nabla\times\mathbf{B}=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rB_\theta\right)-\frac{\partial B_r}{\partial\theta}\right]\hat{\phi}$$

(0.15)
$$=-\frac{2A\sin\theta}{\omega r^3}\left(\left(\frac{1}{kr}-kr\right)\cos\left(kr-\omega t\right)+\sin\left(kr-\omega t\right)\right)\hat{\phi}+\frac{A\sin\theta}{\omega r^3}\left[-kr\cos\left(kr-\omega t\right)-\left(\frac{1}{k}-kr^2-1\right)\sin\left(kr-\omega t\right)\right]\hat{\phi}+\frac{2A\sin\theta}{\omega r^3}\left[\sin\left(kr-\omega t\right)+\frac{\cos\left(kr-\omega t\right)}{kr}\right]\hat{\phi}$$

(0.16)
$$=A\frac{\sin\theta}{r^2c}\left[kr\sin\left(kr-\omega t\right)+\cos\left(kr-\omega t\right)\right]\hat{\phi}$$

where in the last line we used $\omega/k=c$ and collected terms. From 0.1 we get

$$\frac{\partial\mathbf{E}}{\partial t} = A\frac{\sin\theta}{r}\left[\omega\sin\left(kr-\omega t\right)+\frac{\omega\cos\left(kr-\omega t\right)}{kr}\right]\hat{\phi} \tag{0.17}$$

$$= A\frac{\sin\theta}{r^2}\left[krc\sin\left(kr-\omega t\right)+c\cos\left(kr-\omega t\right)\right]\hat{\phi} \tag{0.18}$$

$$= c^2\nabla\times\mathbf{B} \tag{0.19}$$

so the final Maxwell equation is satisfied.

The Poynting vector is, after simplifying

(0.20)

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

$$= \hat{\mathbf{r}}\frac{A^2\sin^2\theta}{\mu_0\omega k r^5}\left[\left(\frac{1}{2}-r^2k^2\right)\sin\left[2\left(kr-\omega t\right)\right]-kr\cos\left[2\left(kr-\omega t\right)\right]+k^3r^3\cos^2\left(kr-\omega t\right)\right]+$$

(0.21)

$$\hat{\theta}\frac{A^2\sin 2\theta}{\mu_0\omega k r^5}\left[\frac{1}{2}\left(k^2r^2-1\right)\sin\left[2\left(kr-\omega t\right)\right]+kr\cos\left[2\left(kr-\omega t\right)\right]\right]$$

The intensity, or time average of the Poynting vector is

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{S}dt \quad (0.22)$$

$$= \frac{A^2k\sin^2\theta}{2\mu_0\omega r^2}\hat{\mathbf{r}} \quad (0.23)$$

The energy flows radially outwards and falls off as r^{-2}.

The total power radiated is the integral of $\mathbf{I}\cdot d\mathbf{a}$ over a sphere:

$$P = \int\mathbf{I}\cdot d\mathbf{a} \quad (0.24)$$

$$= \pi\frac{A^2k}{\mu_0\omega}\int_0^{\pi}\frac{r^2}{r^2}\sin^3\theta d\theta \quad (0.25)$$

$$= \frac{4\pi A^2}{3\mu_0}\frac{k}{\omega} \quad (0.26)$$

$$= \frac{4\pi A^2}{3\mu_0 c} \quad (0.27)$$

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