TRANSMISSION COEFFICIENT FOR A WAVE PASSING THROUGH 3 MEDIA

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.34.

We can extend the analysis of reflection and transmission of waves at a boundary by considering the case of an electromagnetic wave starting out in medium 1 (with wave speed v_1, wave number k_1 and index of refraction $n_1 = c/v_1$), then passing at normal incidence to medium 2 at $z = -d$ and then to medium 3 at $z = 0$. (We've changed the origin from that stated in Griffiths's problem to make the analysis a bit easier, as we'll see). We'd like to find the transmission coefficient between mediums 1 and 3, that is, we'd like to see how much of the wave's energy gets transmitted all the way through the middle medium. We'll assume the mediums are all homogeneous and linear, and that $\mu = \mu_0$ in all of them.

The analysis is much the same as the earlier method, but a bit more complicated. We have a wave $\tilde{\mathbf{E}}_{1R}$ travelling in towards the right in medium 1. At the boundary with medium 2, it gives rise to a reflected wave $\tilde{\mathbf{E}}_{1L}$ travelling to the left in medium 1 and a trasmitted wave $\tilde{\mathbf{E}}_{2R}$ travelling to the right in medium 2. When this wave hits the boundary with medium 3, there is a reflected wave $\tilde{\mathbf{E}}_{2L}$ travelling to the left and a transmitted wave $\tilde{\mathbf{E}}_{3R}$ travelling to the right. There are, of course, corresponding magnetic waves $\tilde{\mathbf{B}}_{1L}$ and so on. We can then apply the boundary conditions to work out the amplitudes. [Actually, the wave reflected back to the left from the 2-3 boundary will hit the 1-2 boundary and be reflected and transmitted there too, so that there is, in principle, an infinite number of reflected and transmitted waves resulting from the wave bouncing back and forth between the two boundaries. However, we can subsume all the left-moving waves into $\tilde{\mathbf{E}}_{1L}$ and $\tilde{\mathbf{E}}_{2L}$ and all the right moving waves into $\tilde{\mathbf{E}}_{1R}$, $\tilde{\mathbf{E}}_{2R}$ and $\tilde{\mathbf{E}}_{3R}$. The important thing is that these waves must satisfy the boundary conditions.]

We can take the electric component to be polarized along the x direction, so that the magnetic component is then along the y direction. The waves are

$$\tilde{\mathbf{E}}_{1L} = E_{1L}e^{i(-k_1z-\omega t)}\hat{\mathbf{x}} \tag{1}$$
$$\tilde{\mathbf{E}}_{1R} = E_{1R}e^{i(k_1z-\omega t)}\hat{\mathbf{x}} \tag{2}$$
$$\tilde{\mathbf{E}}_{2L} = E_{2L}e^{i(-k_2z-\omega t)}\hat{\mathbf{x}} \tag{3}$$
$$\tilde{\mathbf{E}}_{2R} = E_{2R}e^{i(-k_2z-\omega t)}\hat{\mathbf{x}} \tag{4}$$
$$\tilde{\mathbf{E}}_{3R} = E_{3R}e^{i(k_3z-\omega t)}\hat{\mathbf{x}} \tag{5}$$
$$\tilde{\mathbf{B}}_{1L} = -\frac{1}{v_1}E_{1L}e^{i(-k_1z-\omega t)}\hat{\mathbf{y}} \tag{6}$$
$$\tilde{\mathbf{B}}_{1R} = \frac{1}{v_1}E_{1R}e^{i(k_1z-\omega t)}\hat{\mathbf{y}} \tag{7}$$
$$\tilde{\mathbf{B}}_{2L} = -\frac{1}{v_2}E_{2L}e^{i(-k_2z-\omega t)}\hat{\mathbf{y}} \tag{8}$$
$$\tilde{\mathbf{B}}_{2R} = \frac{1}{v_2}E_{2R}e^{i(k_2z-\omega t)}\hat{\mathbf{y}} \tag{9}$$
$$\tilde{\mathbf{B}}_{3R} = \frac{1}{v_3}E_{3R}e^{i(k_3z-\omega t)}\hat{\mathbf{y}} \tag{10}$$

The negative signs for the left-moving magnetic waves are to keep the Poynting vector pointing to the left. The coefficients E_{1L} and so on are actually complex numbers, but we've dropped the tilde (and the subscript 0 that Griffiths uses) to make the notation simpler.

Because the medium 2-3 boundary consists of an incident right-moving wave, a reflected left-moving wave and a transmitted wave, it is identical to the case we treated earlier, provided we take $z = 0$ at this point (which we've done). We can therefore write down the results:

$$E_{2L} = \frac{v_3 - v_2}{v_2 + v_3}E_{2R} \tag{11}$$
$$E_{3R} = \frac{2v_3}{v_2 + v_3}E_{2R} \tag{12}$$

Now for the medium 1-2 boundary at $z = -d$. From the boundary condition $\mathbf{E}_1^{\|} = \mathbf{E}_2^{\|}$ and (since $\mu = \mu_0$ everywhere) $\mathbf{B}_1^{\|} = \mathbf{B}_2^{\|}$ we get

$$E_{2R}e^{-ik_2d} + E_{2L}e^{ik_2d} = E_{1R}e^{-ik_1d} + E_{1L}e^{ik_1d} \tag{13}$$
$$\frac{1}{v_2}\left(E_{2R}e^{-ik_2d} - E_{2L}e^{ik_2d}\right) = \frac{1}{v_1}\left(E_{1R}e^{-ik_1d} - E_{1L}e^{ik_1d}\right) \tag{14}$$

These 4 equations are linear in the E coefficients so it's straightforward (although tedious) to solve them. It's easiest to let Maple handle this part, and we get (since we're interested only in expressing E_{3R} in terms of E_{1R}):

(15)
$$E_{1R} = E_{3R}\frac{e^{ik_1 d}}{4v_2v_3}\left[e^{ik_2 d}\left(v_2v_3+v_1v_2-v_2^2-v_1v_3\right)+e^{-ik_2 d}\left(v_2v_3+v_1v_2+v_2^2+v_1v_3\right)\right]$$
(16)
$$= E_{3R}\frac{e^{ik_1 d}}{4v_2v_3}\left[2\cos(k_2d)(v_2v_3+v_1v_2)-2i\sin(k_2d)\left(v_2^2+v_1v_3\right)\right]$$

The intensity of a wave is

$$I = \frac{|E|^2}{2\mu v} \tag{17}$$

and the transmission coefficient is

$$T = \frac{I_{3R}}{I_{1R}} \tag{18}$$

so we get

(19)
$$T^{-1} = \frac{v_3}{v_1}\frac{4\cos^2(k_2d)(v_2v_3+v_1v_2)^2+4\sin^2(k_2d)\left(v_2^2+v_1v_3\right)^2}{16v_2^2v_3^2}$$
(20)
$$= \frac{1}{4v_1v_3}\left[\frac{(v_2v_3+v_1v_2)^2}{v_2^2}\left(1-\sin^2(k_2d)\right)+\frac{\left(v_2^2+v_1v_3\right)^2}{v_2^2}\sin^2(k_2d)\right]$$
(21)
$$= \frac{1}{4v_1v_3}\left[(v_3+v_1)^2+\sin^2(k_2d)\frac{\left(v_2^2+v_1v_3\right)^2-(v_2v_3+v_1v_2)^2}{v_2^2}\right]$$
(22)
$$= \frac{1}{4v_1v_3}\left[(v_3+v_1)^2+\sin^2(k_2d)\frac{\left(v_1^2-v_2^2\right)\left(v_3^2-v_2^2\right)}{v_2^2}\right]$$

We can express this in terms of the indexes of refraction by noting that since $n_i = c/v_i$ and the power of the vs is the same in numerator and denominator so the factors of c cancel out and we can replace v_i with $1/n_i$. We can then multiply the first term top and bottom by $n_1^2 n_3^2$ and the second term top and bottom by $n_1^2 n_3^2 n_2^4$ to get

$$T^{-1}=\frac{1}{4n_1n_3}\left[(n_1+n_3)^2+\sin^2(k_2d)\frac{\left(n_1^2-n_2^2\right)\left(n_3^2-n_2^2\right)}{n_2^2}\right] \tag{23}$$

Finally we note that the wave speed in medium 2 is $v_2=\omega/k_2=c/n_2$ so $k_2=n_2\omega/c$ and we get

$$T^{-1}=\frac{1}{4n_1n_3}\left[(n_1+n_3)^2+\sin^2\left(\frac{n_2\omega d}{c}\right)\frac{\left(n_1^2-n_2^2\right)\left(n_3^2-n_2^2\right)}{n_2^2}\right] \tag{24}$$

COMMENTS

From: Paul Landini
Time: October 22, 2017 at 7:30 pm
Comment: In problem 9.34 of Griffiths' Electrodynamics book, how do you arrive at equation 0.19 based from 0.16? I do not see where the e^(ik_1d) disappears at, nor do I see where the imaginary terms go in the expansion of the squared terms.
Thanks! Paul
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Equation 19 is the ratio of two intensities, which are defined by equation 17, which takes the square modulus of E. The square modulus of a complex number $a+ib$ is a^2+b^2, which is why the i disappears. Also $\left|e^{ix}\right|=1$ for any real x.

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