

TOTAL INTERNAL REFLECTION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.37.

When we looked at the behaviour of waves passing from one medium to another at an angle, one of the consequences was Snell's law of refraction which says

$$(1) \quad \frac{\sin \theta_I}{\sin \theta_T} = \frac{n_2}{n_1}$$

where θ_I is the angle of incidence (angle between the wave vector and the normal to the surface) in medium 1 with index of refraction n_1 , and θ_T is the angle of the refracted wave in medium 2. If $n_1 > n_2$, that is, the wave is incident from a medium (such as water) with a higher index of refraction than medium 2 (such as air), then we can reach a critical incident angle θ_c where the refracted angle is $\pi/2$, so that the refracted wave moves parallel to the interface. This happens when

$$(2) \quad \sin \theta_c = \frac{n_2}{n_1}$$

If $\theta_I > \theta_c$, no wave is transmitted through the interface and the entire wave is reflected back into medium 1. This is known as *total internal reflection*.

To see what happens in this case, we can follow through the same derivation as before, except we allow the wave vector \mathbf{k}_T of the transmitted wave to be complex. That is, for a given θ_T we say that (assuming that the incident and transmitted plane is the xz plane):

$$(3) \quad \mathbf{k}_T = k_T (\sin \theta_T \hat{\mathbf{x}} + \cos \theta_T \hat{\mathbf{z}})$$

The wave vector has, as usual, the magnitude of

$$(4) \quad k_T = \frac{\omega}{v_2} = \frac{\omega n_2}{c}$$

Now suppose $\theta_I > \theta_c$. In that case,

$$(5) \quad \sin \theta_T = \frac{n_1}{n_2} \sin \theta_I > \frac{n_1 n_2}{n_2 n_1} = 1$$

$$(6) \quad \cos \theta_T = \sqrt{1 - \sin^2 \theta_T}$$

$$(7) \quad = i\sqrt{\sin^2 \theta_T - 1}$$

$$(8) \quad = i\sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_I - 1}$$

$$(9) \quad \mathbf{k}_T = k_T \left(\frac{n_1}{n_2} \sin \theta_I \hat{\mathbf{x}} + i\sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_I - 1} \hat{\mathbf{z}} \right)$$

$$(10) \quad = \frac{\omega n_2}{c} \left(\frac{n_1}{n_2} \sin \theta_I \hat{\mathbf{x}} + i\sqrt{\frac{n_1^2}{n_2^2} \sin^2 \theta_I - 1} \hat{\mathbf{z}} \right)$$

$$(11) \quad = k\hat{\mathbf{x}} + i\kappa\hat{\mathbf{z}}$$

where

$$(12) \quad k \equiv \frac{\omega n_1}{c} \sin \theta_I$$

$$(13) \quad \kappa \equiv \frac{\omega}{c} \sqrt{n_1^2 \sin^2 \theta_I - n_2^2}$$

That is, we venture into the realm of complex variables, and θ_T can no longer be interpreted as a geometric angle. However, let's proceed with the analysis and see what happens.

First, we look at the electric field, which has the general form

$$(14) \quad \tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}$$

Plugging in 11, we get

$$(15) \quad \tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0T} e^{-\kappa z} e^{i(kx - \omega t)}$$

That is, the transmitted wave propagates in the x direction (parallel to the interface) and is attenuated in the z direction.

How much of the wave is reflected in this case? For a wave with polarization parallel to the incident plane (that is, \mathbf{E} has only an x component), we found earlier That the reflected amplitude is

$$(16) \quad E_R = \frac{\alpha - \beta}{\alpha + \beta} E_I$$

where

$$(17) \quad \alpha \equiv \frac{\cos \theta_T}{\cos \theta_I}$$

$$(18) \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

In this case, α is purely imaginary and β is real, so the reflection coefficient is

$$(19) \quad R = \left| \frac{E_R}{E_I} \right|^2 = \left| \frac{\alpha - \beta}{\alpha + \beta} \right|^2 = 1$$

since

$$(20) \quad |\alpha - \beta|^2 = |\alpha|^2 + \beta^2 = |\alpha + \beta|^2$$

For perpendicular polarization, we have

$$(21) \quad E_R = \frac{1 - \alpha\beta}{1 + \alpha\beta} E_I$$

and again, since 1 is real and $\alpha\beta$ is purely imaginary

$$(22) \quad R = \left| \frac{1 - \alpha\beta}{1 + \alpha\beta} \right|^2 = 1$$

Thus the reflection is indeed total for both polarizations.

Still with perpendicular polarization, the electric field is entirely in the y direction so

$$(23) \quad \tilde{\mathbf{E}}_T = \tilde{E}_0 e^{-\kappa z} e^{i(kx - \omega t)} \hat{\mathbf{y}}$$

The magnetic field is given by (using 11 and $\mathbf{k}_T = \frac{\omega n_2}{c} \hat{\mathbf{k}}_T$ and $v_2 = c/n_2$)

$$\begin{aligned}
(24) \quad \tilde{\mathbf{B}}_T &= \frac{1}{v_2} \hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T \\
(25) \quad &= \frac{1}{v_2} \frac{c}{\omega n_2} \tilde{E}_0 e^{-\kappa z} e^{i(kx - \omega t)} (k\hat{\mathbf{z}} - i\kappa\hat{\mathbf{x}}) \\
(26) \quad &= \frac{\tilde{E}_0}{\omega} e^{-\kappa z} e^{i(kx - \omega t)} (k\hat{\mathbf{z}} - i\kappa\hat{\mathbf{x}})
\end{aligned}$$

Taking the real parts to get the actual fields, we have

$$\begin{aligned}
(27) \quad \mathbf{E}_T &= E_0 e^{-\kappa z} \cos(kx - \omega t) \hat{\mathbf{y}} \\
(28) \quad \mathbf{B}_T &= \frac{E_0}{\omega} e^{-\kappa z} (k \cos(kx - \omega t) \hat{\mathbf{z}} + \kappa \sin(kx - \omega t) \hat{\mathbf{x}})
\end{aligned}$$

To check that these fields satisfy Maxwell's equations requires grinding away at some derivatives, which we can do in Maple. The divergences are fairly easy and we get

$$(29) \quad \nabla \cdot \mathbf{E}_T = \nabla \cdot \mathbf{B}_T = 0$$

The curls give

$$(30) \quad \nabla \times \mathbf{E}_T = E_0 e^{-\kappa z} (-k \sin(kx - \omega t) \hat{\mathbf{z}} + \kappa \cos(kx - \omega t) \hat{\mathbf{x}})$$

$$(31) \quad = -\frac{\partial \mathbf{B}_T}{\partial t}$$

$$(32) \quad \nabla \times \mathbf{B}_T = \frac{E_0}{\omega} e^{-\kappa z} \sin(kx - \omega t) \hat{\mathbf{y}} (k^2 - \kappa^2)$$

$$(33) \quad = \frac{n_2^2}{c^2} \omega E_0 e^{-\kappa z} \sin(kx - \omega t) \hat{\mathbf{y}}$$

$$(34) \quad = \frac{1}{v_2^2} \frac{\partial \mathbf{E}_T}{\partial t}$$

Thus all four of Maxwell's equations are satisfied.

The Poynting vector is

$$(35) \quad \mathbf{S} = \frac{1}{\mu} \mathbf{E}_T \times \mathbf{B}_T$$

$$(36) \quad = \frac{E_0^2 e^{-2\kappa z}}{\mu \omega} (k \cos^2(kx - \omega t) \hat{\mathbf{x}} - \kappa \sin(kx - \omega t) \cos(kx - \omega t) \hat{\mathbf{z}})$$

Integrating this over one cycle ($t = 0$ to $2\pi/\omega$) gives zero for the z component, thus no energy is transmitted perpendicular to the interface, and all the energy flows in the x direction, parallel to the interface.