TOTAL INTERNAL REFLECTION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.37.

When we looked at the behaviour of waves passing from one medium to another at an angle, one of the consequences was Snell's law of refraction which says

$$\frac{\sin\theta_I}{\sin\theta_T}=\frac{n_2}{n_1} \tag{1}$$

where θ_I is the angle of incidence (angle between the wave vector and the normal to the surface) in medium 1 with index of refraction n_1, and θ_T is the angle of the refracted wave in medium 2. If $n_1>n_2$, that is, the wave is incident from a medium (such as water) with a higher index of refraction than medium 2 (such as air), then we can reach a critical incident angle θ_c where the refracted angle is $\pi/2$, so that the refracted wave moves parallel to the interface. This happens when

$$\sin\theta_c=\frac{n_2}{n_1} \tag{2}$$

If $\theta_I>\theta_c$, no wave is transmitted through the interface and the entire wave is reflected back into medium 1. This is known as *total internal reflection*.

To see what happens in this case, we can follow through the same derivation as before, except we allow the wave vector $\mathbf{k}_T$ of the transmitted wave to be complex. That is, for a given θ_T we say that (assuming that the incident and transmitted plane is the xz plane):

$$\mathbf{k}_T=k_T\left(\sin\theta_T\hat{\mathbf{x}}+\cos\theta_T\hat{\mathbf{z}}\right) \tag{3}$$

The wave vector has, as usual, the magnitude of

$$k_T=\frac{\omega}{v_2}=\frac{\omega n_2}{c} \tag{4}$$

Now suppose $\theta_I>\theta_c$. In that case,

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > \frac{n_1}{n_2}\frac{n_2}{n_1} = 1 \tag{5}$$

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} \tag{6}$$

$$= i\sqrt{\sin^2\theta_T - 1} \tag{7}$$

$$= i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1} \tag{8}$$

$$\mathbf{k}_T = k_T\left(\frac{n_1}{n_2}\sin\theta_I\hat{\mathbf{x}} + i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1\hat{\mathbf{z}}}\right) \tag{9}$$

$$= \frac{\omega n_2}{c}\left(\frac{n_1}{n_2}\sin\theta_I\hat{\mathbf{x}} + i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1\hat{\mathbf{z}}}\right) \tag{10}$$

$$= k\hat{\mathbf{x}} + i\kappa\hat{\mathbf{z}} \tag{11}$$

where

$$k \equiv \frac{\omega n_1}{c}\sin\theta_I \tag{12}$$

$$\kappa \equiv \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2} \tag{13}$$

That is, we venture into the realm of complex variables, and θ_T can no longer be interpreted as a geometric angle. However, let's proceed with the analysis and see what happens.

First, we look at the electric field, which has the general form

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)} \tag{14}$$

Plugging in 11, we get

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T}e^{-\kappa z}e^{i(kx-\omega t)} \tag{15}$$

That is, the transmitted wave propagates in the x direction (parallel to the interface) and is attenuated in the z direction.

How much of the wave is reflected in this case? For a wave with polarization parallel to the incident plane (that is, $\mathbf{E}$ has only an x component), we found earlier That the reflected amplitude is

$$E_R = \frac{\alpha-\beta}{\alpha+\beta}E_I \tag{16}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \tag{17}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{18}$$

In this case, α is purely imaginary and β is real, so the reflection coefficient is

$$R = \left|\frac{E_R}{E_I}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right| = 1 \tag{19}$$

since

$$|\alpha-\beta|^2 = |\alpha|^2 + \beta^2 = |\alpha+\beta|^2 \tag{20}$$

For perpendicular polarization, we have

$$E_R = \frac{1-\alpha\beta}{1+\alpha\beta} E_I \tag{21}$$

and again, since 1 is real and $\alpha\beta$ is purely imaginary

$$R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = 1 \tag{22}$$

Thus the reflection is indeed total for both polarizations.

Still with perpendicular polarization, the electric field is entirely in the y direction so

$$\tilde{\mathbf{E}}_T = \tilde{E}_0 e^{-\kappa z} e^{i(kx-\omega t)} \hat{\mathbf{y}} \tag{23}$$

The magnetic field is given by (using 11 and $\mathbf{k}_T = \frac{\omega n_2}{c}\hat{\mathbf{k}}_T$ and $v_2 = c/n_2$)

$$\tilde{\mathbf{B}}_T = \frac{1}{v_2}\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T \tag{24}$$

$$= \frac{1}{v_2}\frac{c}{\omega n_2}\tilde{E}_0 e^{-\kappa z} e^{i(kx-\omega t)} \left(k\hat{\mathbf{z}} - i\kappa\hat{\mathbf{x}}\right) \tag{25}$$

$$= \frac{\tilde{E}_0}{\omega} e^{-\kappa z} e^{i(kx-\omega t)} \left(k\hat{\mathbf{z}} - i\kappa\hat{\mathbf{x}}\right) \tag{26}$$

Taking the real parts to get the actual fields, we have

$$\mathbf{E}_T = E_0 e^{-\kappa z}\cos(kx-\omega t)\hat{\mathbf{y}} \tag{27}$$

$$\mathbf{B}_T = \frac{E_0}{\omega}e^{-\kappa z}\left(k\cos(kx-\omega t)\hat{\mathbf{z}}+\kappa\sin(kx-\omega t)\hat{\mathbf{x}}\right) \tag{28}$$

To check that these fields satisfy Maxwell's equations requires grinding away at some derivatives, which we can do in Maple. The divergences are fairly easy and we get

$$\nabla\cdot\mathbf{E}_T = \nabla\cdot\mathbf{B}_T = 0 \tag{29}$$

The curls give

$$\nabla\times\mathbf{E}_T = E_0 e^{-\kappa z}\left(-k\sin(kx-\omega t)\hat{\mathbf{z}}+\kappa\cos(kx-\omega t)\hat{\mathbf{x}}\right) \tag{30}$$

$$= -\frac{\partial\mathbf{B}_T}{\partial t} \tag{31}$$

$$\nabla\times\mathbf{B}_T = \frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)\hat{\mathbf{y}}\left(k^2-\kappa^2\right) \tag{32}$$

$$= \frac{n_2^2}{c^2}\omega E_0 e^{-\kappa z}\sin(kx-\omega t)\hat{\mathbf{y}} \tag{33}$$

$$= \frac{1}{v_2^2}\frac{\partial\mathbf{E}_T}{\partial t} \tag{34}$$

Thus all four of Maxwell's equations are satisfied.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu}\mathbf{E}_T\times\mathbf{B}_T \tag{35}$$

$$= \frac{E_0^2 e^{-2\kappa z}}{\mu\omega}\left(k\cos^2(kx-\omega t)\hat{\mathbf{x}}-\kappa\sin(kx-\omega t)\cos(kx-\omega t)\hat{\mathbf{z}}\right) \tag{36}$$

Integrating this over one cycle ($t=0$ to $2\pi/\omega$) gives zero for the z component, thus no energy is transmitted perpendicular to the interface, and all the energy flows in the x direction, parallel to the interface.