RECTANGULAR RESONANT CAVITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 9.38.

A wave guide consisting of a completely enclosed volume is known as a *resonant cavity*. The simplest resonant cavity is created by taking a rectangular wave guide and closing off the ends to form a rectangular box with dimensions of a in the x direction, b in the y direction and d in the z direction. To find the fields, we can't just assume that the wave propagates in the z direction with the same z dependence for all three components of each field, that is, we can't take the waves to be

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0(x,y)\,e^{i(kz-\omega t)} \tag{1}$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0(x,y)\,e^{i(kz-\omega t)} \tag{2}$$

This time, there has to be an explicit dependence on z in $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$, so we take the waves to be

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0(x,y,z)\,e^{-i\omega t} \tag{3}$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0(x,y,z)\,e^{-i\omega t} \tag{4}$$

We can apply the two curl Maxwell equations to get

$$\nabla\times\tilde{\mathbf{E}} = -\frac{\partial\tilde{\mathbf{B}}}{\partial t} \tag{5}$$

$$= i\omega\tilde{\mathbf{B}} \tag{6}$$

$$\nabla\times\tilde{\mathbf{B}} = \frac{1}{c^2}\frac{\partial\tilde{\mathbf{E}}}{\partial t} \tag{7}$$

$$= -i\frac{\omega}{c^2}\tilde{\mathbf{E}} \tag{8}$$

As the curl affects only the spatial coordinates, we can cancel off $e^{-i\omega t}$ from both sides of these equations to get

$$\nabla\times\tilde{\mathbf{E}}_0 = i\omega\tilde{\mathbf{B}}_0 \tag{9}$$

$$\nabla\times\tilde{\mathbf{B}}_0 = -i\frac{\omega}{c^2}\tilde{\mathbf{E}}_0 \tag{10}$$

Taking the curl of the first of these equations, we get

$$\begin{aligned}\nabla\times\left(\nabla\times\tilde{\mathbf{E}}_0\right) &= \nabla\times\left(\nabla\cdot\tilde{\mathbf{E}}_0\right)-\nabla^2\tilde{\mathbf{E}}_0 & (11)\\ &= -\nabla^2\tilde{\mathbf{E}}_0 & (12)\\ &= i\omega\nabla\times\tilde{\mathbf{B}}_0 & (13)\\ &= \frac{\omega^2}{c^2}\tilde{\mathbf{E}}_0 & (14)\end{aligned}$$

Therefore we get

$$\nabla^2\tilde{\mathbf{E}}_0=-\frac{\omega^2}{c^2}\tilde{\mathbf{E}}_0 \tag{15}$$

giving the same differential equation for each component. We can use separation of variables to solve them. For the x component, let $E_x = X(x)Y(y)Z(z)$. Then

$$\begin{aligned}X''YZ+XY''Z+XYZ'' &= -\frac{\omega^2}{c^2}XYZ & (16)\\ \frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z} &= -\frac{\omega^2}{c^2} & (17)\end{aligned}$$

As usual, each term on the LHS must be equal to a constant, and the solution of the resulting 3 equations is a sum of a sine and a cosine, so we have

$$\begin{aligned}X(x) &= A\sin k_{xx}x+H\cos k_{xx}x & (18)\\ Y(y) &= C\sin k_{xy}y+D\cos k_{xy}y & (19)\\ Z(z) &= F\sin k_{xz}z+G\cos k_{xz}z & (20)\end{aligned}$$

with similar equations for the other components of **E**. The double subscript such as k_{xy} means that this k belongs to E_x in the function $Y(y)$.

The boundary conditions are

$$\begin{aligned}\mathbf{E}^{\parallel} &= 0 & (21)\\ B^{\perp} &= 0 & (22)\end{aligned}$$

at all boundaries, so we can use this to constrain the k_is. E_x must be zero at $y=0,b$ and $z=0,d$ which means

$$D = G = 0 \quad (23)$$
$$k_{xy} = \frac{n\pi}{b} \quad (24)$$
$$k_{xz} = \frac{\ell\pi}{d} \quad (25)$$

We can't, at this stage, put any constraints on k_{xx} since it doesn't figure in any of the boundary conditions. Putting it all together, and condensing the constants into $X(x)$, we get

$$E_x = (A\sin k_{xx}x + H\cos k_{xx}x)\sin\frac{n\pi}{b}y\sin\frac{\ell\pi}{d}z \quad (26)$$

For E_y we get the same solutions 18, 19 and 20 as above. E_y must be zero at $x = 0, a$ and $z = 0, d$ so we get

$$H = G = 0 \quad (27)$$
$$k_{yx} = \frac{m\pi}{a} \quad (28)$$
$$k_{yz} = \frac{\ell\pi}{d} \quad (29)$$

This gives

$$E_y = (C\sin k_{yy}y + D\cos k_{yy}y)\sin\frac{m\pi}{a}x\sin\frac{\ell\pi}{d}z \quad (30)$$

Finally, for E_z the boundary conditions require it to be zero at $x = 0, a$ and $y = 0, b$ so we get

$$H = D = 0 \quad (31)$$
$$k_{zx} = \frac{m\pi}{a} \quad (32)$$
$$k_{zy} = \frac{n\pi}{b} \quad (33)$$

This gives

$$E_z = (F\sin k_{zz}z + G\cos k_{zz}z)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad (34)$$

Now we can invoke Gauss's law in vacuum, which states that $\nabla\cdot\mathbf{E} = 0$. This gives

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= k_{xx}\left(A\cos k_{xx}x - H\sin k_{xx}x\right)\sin\frac{n\pi}{b}y\sin\frac{\ell\pi}{d}z + \qquad (35)\\
&\quad k_{yy}\left(C\cos k_{yy}y - D\sin k_{yy}y\right)\sin\frac{m\pi}{a}x\sin\frac{\ell\pi}{d}z + \\
&\quad k_{zz}\left(F\cos k_{zz}z - G\sin k_{zz}z\right)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \\
&= 0 \qquad (36)
\end{aligned}$$

This equation must be true for all values of (x,y,z) so if we choose $x=0$ we get $A=0$, or if $y=0$ then $C=0$, or if $z=0$ then $F=0$, so we have

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= -k_{xx}H\sin k_{xx}x\sin\frac{n\pi}{b}y\sin\frac{\ell\pi}{d}z + \qquad (37)\\
&\quad -k_{yy}D\sin k_{yy}y\sin\frac{m\pi}{a}x\sin\frac{\ell\pi}{d}z + \\
&\quad -k_{zz}G\sin k_{zz}z\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \\
&= 0 \qquad (38)
\end{aligned}$$

We should now be able to conclude that arguments of the sine functions for each variable are equal, that is, that $k_{xx}=\frac{m\pi}{a}$ and so on. I haven't been able to find a proof of this, although it seems to have something to do with Fourier analysis. The argument is along the lines of: the only way an expansion of sines can be zero for all points is if they are all the same sine and they add up to zero identically. Given that, we get

$$-\left(\frac{m\pi}{a}H+\frac{n\pi}{b}D+\frac{\ell\pi}{d}G\right)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\sin\frac{\ell\pi}{d}z = 0 \qquad (39)$$

$$\frac{m\pi}{a}H+\frac{n\pi}{b}D+\frac{\ell\pi}{d}G = 0 \qquad (40)$$

We can therefore simplify the notation by defining

$$\begin{aligned}
k_x &= \frac{m\pi}{a} \qquad (41)\\
k_y &= \frac{n\pi}{b} \qquad (42)\\
k_z &= \frac{\ell\pi}{d} \qquad (43)
\end{aligned}$$

So the electric field is

$$\tilde{\mathbf{E}} = He^{-i\omega t}\cos k_x x \sin k_y y \sin k_z z\hat{\mathbf{x}} + De^{-i\omega t}\sin k_x x \cos k_y y \sin k_z z\hat{\mathbf{y}} + \\ Ge^{-i\omega t}\sin k_x x \sin k_y y \cos k_z z\hat{\mathbf{z}} \tag{44}$$

The magnetic field can be found from the Maxwell equation

$$\begin{aligned}
\nabla \times \tilde{\mathbf{E}} &= -\frac{\partial \tilde{\mathbf{B}}}{\partial t} &(45)\\
&= (Gk_y - Dk_z)\, e^{-i\omega t}\sin k_x x \cos k_y y \cos k_z z\hat{\mathbf{x}} + &(46)\\
&\quad (Hk_z - Gk_x)\, e^{-i\omega t}\cos k_x x \sin k_y y \cos k_z z\hat{\mathbf{y}} + \\
&\quad (Dk_x - Hk_y)\, e^{-i\omega t}\sin k_x x \cos k_y y \cos k_z z\hat{\mathbf{z}}
\end{aligned}$$

Integrating with respect to time gives

$$\begin{aligned}
\tilde{\mathbf{B}} &= -\frac{i}{\omega}(Gk_y - Dk_z)\, e^{-i\omega t}\sin k_x x \cos k_y y \cos k_z z\hat{\mathbf{x}} - &(47)\\
&\quad \frac{i}{\omega}(Hk_z - Gk_x)\, e^{-i\omega t}\cos k_x x \sin k_y y \cos k_z z\hat{\mathbf{y}} - \\
&\quad \frac{i}{\omega}(Dk_x - Hk_y)\, e^{-i\omega t}\sin k_x x \cos k_y y \cos k_z z\hat{\mathbf{z}}
\end{aligned}$$

In all cases, 17 requires that

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \tag{48}$$

so the resonant frequency for mode $mn\ell$ is

$$\begin{aligned}
\omega_{mn\ell} &= c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2} &(49)\\
&= c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{\ell}{d}\right)^2} &(50)
\end{aligned}$$