MAXWELL'S EQUATIONS IN TERMS OF POTENTIALS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.1.

Maxwell's equations are

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

The are completely general, in that they describe the interactions between electric and magnetic fields and the charges and currents that generate them, in both the static (fixed charge and current densities) and dynamic (time-varying) cases. In the static case, the two time derivatives are zero, which means that $\nabla\times\mathbf{E}=0$. Since any vector field whose curl is zero can be represented as the gradient of a scalar field, we could write $\mathbf{E}=-\nabla V$ where V is the potential. The presence of the term $-\frac{\partial\mathbf{B}}{\partial t}$, however, means that we can't, in general, write $\mathbf{E}$ as the gradient of a scalar field in the dynamic case.

Because $\nabla\cdot\mathbf{B}=0$ we can write $\mathbf{B}$ as the curl of a vector field

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{5}$$

This is still true in the dynamic case. Substituting this into 3 we get

$$\nabla\times\mathbf{E} = -\frac{\partial\nabla\times\mathbf{A}}{\partial t} \tag{6}$$

$$\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right) = 0 \tag{7}$$

The LHS is the curl of a vector field that combines the electric field and the magnetic vector potential, and since it is zero even in the dynamic case, we can write this vector field as the gradient of a scalar field V. [We should be able to use the same symbol V for the potential in the dynamic case, since if $\frac{\partial \mathbf{A}}{\partial t}=0$ we regain the static equations $\nabla\times\mathbf{E}=0$ and $\mathbf{E}=-\nabla V$.] That is, we can write

$$\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{8}$$

$$\mathbf{E} = -\nabla V-\frac{\partial \mathbf{A}}{\partial t} \tag{9}$$

Equations 5 and 9 effectively replace 2 and 3. Using 5 and 9 we can eliminate $\mathbf{E}$ and $\mathbf{B}$ from 1 and 4:

$$\nabla\cdot\mathbf{E}=-\nabla^2 V-\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \tag{10}$$

$$\nabla^2 V+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-\frac{\rho}{\varepsilon_0} \tag{11}$$

$$\nabla\times\mathbf{B}=\nabla\times(\nabla\times\mathbf{A}) \tag{12}$$

$$=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{13}$$

$$\nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{J}-\mu_0\varepsilon_0\nabla\frac{\partial V}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{14}$$

We can rearrange the last equation using the identity

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} \tag{15}$$

so we get

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\mu_0\varepsilon_0\nabla\frac{\partial V}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{16}$$

$$\nabla^2\mathbf{A}-\mu_0\varepsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\mathbf{J} \tag{17}$$

To summarize:

$$\boxed{\nabla^2 V+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-\frac{\rho}{\varepsilon_0}} \tag{18}$$

$$\boxed{\nabla^2\mathbf{A}-\mu_0\varepsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\mathbf{J}} \tag{19}$$

These two equations comprise 4 equations (one from 18 and one for each vector component in 19) for four functions (V and $\mathbf{A}$), and their solution allows us to calculate both $\mathbf{E}$ and $\mathbf{B}$ by means of 9 and 5, so they form a complete replacement for the original set of 4 Maxwell equations that we started with. This reduces the number of equations to be solved from 6 (for the 3 components of each of $\mathbf{E}$ and $\mathbf{B}$) to 4.

We can write these equations in a more compact form using the d'Alembertian operator

$$\square^2 \equiv \nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} \tag{20}$$

and defining the function

$$L \equiv \nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} \tag{21}$$

We get

$$\square^2 V = \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} \tag{22}$$

$$= \nabla^2 V - \frac{\partial L}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \tag{23}$$

$$\square^2 V + \frac{\partial L}{\partial t} = \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \tag{24}$$

$$= -\frac{\rho}{\varepsilon_0} \tag{25}$$

using 18 in the last line. Also,

$$\square^2 \mathbf{A} = \nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{26}$$

$$\square^2 \mathbf{A} - \nabla L = -\mu_0 \mathbf{J} \tag{27}$$

using 19. In summary

$$\boxed{\square^2 V + \frac{\partial L}{\partial t} = -\frac{\rho}{\varepsilon_0}} \tag{28}$$

$$\boxed{\square^2 \mathbf{A} - \nabla L = -\mu_0 \mathbf{J}} \tag{29}$$

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