

ENERGY FLOW IN TIME-DEPENDENT FIELDS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.2.

Suppose we have time-dependent electric and magnetic potentials given by

$$V = 0 \quad (1)$$

$$\mathbf{A} = \begin{cases} \frac{\mu_0 k}{4c} (ct - |x|)^2 \hat{\mathbf{z}} & \text{for } |x| < ct \\ 0 & \text{for } |x| > ct \end{cases} \quad (2)$$

where k is a constant and c is the speed of light.

We can get the fields by using our earlier formulas

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (3)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (4)$$

We get, for $|x| < ct$

$$\mathbf{E} = \frac{\mu_0 k}{2} (ct - |x|) \quad (5)$$

$$\mathbf{B} = -\frac{\mu_0 k}{4c} \frac{\partial}{\partial x} (ct - |x|)^2 \hat{\mathbf{y}} \quad (6)$$

$$= \pm \frac{\mu_0 k}{2c} (ct - |x|) \hat{\mathbf{y}} \quad (7)$$

where in the last equation the $+$ applies to $x > 0$ and the $-$ to $x < 0$. For $|x| > ct$, all fields are zero.

Griffiths shows in his example 10.1 that these fields satisfy Maxwell's equations and arise from a surface current flowing in the $x = 0$ plane given by

$$\mathbf{K} = kt \hat{\mathbf{z}} \quad (8)$$

for $t \geq 0$. That is, the current starts at $t = 0$ and its effects travel outwards

in the x direction at the speed c . What we'll do here is calculate the energy flow due to this current. We'll consider a rectangular box of width w in the y direction and length ℓ in the z direction extending from $x = d$ to $x = d + h$. At time $t_1 = d/c$ the energy is zero inside the box since the fields from the current have only just reached the lower face of the box at $x = d$. At time $t_2 = (d + h)/c$, the fields have reached the outer face of the box so the entire box is filled with fields. At this time the energy within the box can be calculated from the energy density

$$U = \frac{1}{2} \left[\epsilon_0 \int E^2 d^3\mathbf{r} + \frac{1}{\mu_0} \int B^2 d^3\mathbf{r} \right] \quad (9)$$

$$= \frac{w\ell}{2} \left[\epsilon_0 \left(\frac{\mu_0 k}{2} \right)^2 \int_d^{d+h} (d+h-x)^2 dx + \frac{1}{\mu_0} \left(\frac{\mu_0 k}{2c} \right)^2 \int_d^{d+h} (d+h-x)^2 dx \right] \quad (10)$$

$$= \frac{w\ell\mu_0 k^2}{4c^2} \int_d^{d+h} (d+h-x)^2 dx \quad (11)$$

$$= \frac{w\ell\mu_0 k^2}{4c^2} \int_0^h v^2 dv \quad (12)$$

$$= \frac{w\ell\mu_0 k^2 h^3}{12c^2} \quad (13)$$

where in the second line we used $\epsilon_0\mu_0 = 1/c^2$ and in the third line we used the substitution $v = d + h - x$ in the integral.

Now we can calculate the flow of energy into the box as a function of time using the Poynting vector which gives the rate per unit area at which energy crosses a surface. We have for the surface $x = d$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (14)$$

$$= \frac{1}{\mu_0} \frac{\mu_0^2 k^2}{4c} (ct - d)^2 \quad (15)$$

The total rate at which energy flows into the box via this face is thus

$$\frac{dU}{dt} = \frac{w\ell\mu_0 k^2}{4c} (ct - d)^2 \quad (16)$$

and the total energy that flows into the box between times $t_1 = d/c$ and $t_2 = (d + h)/c$ is

$$U = \frac{w\ell\mu_0k^2}{4c} \int_{d/c}^{(d+h)/c} (ct-d)^2 dt \quad (17)$$

$$= \frac{w\ell\mu_0k^2}{4c^2} \int_0^h v^2 dv \quad (18)$$

$$= \frac{w\ell\mu_0k^2h^3}{12c^2} \quad (19)$$

which of course agrees with our earlier calculation of the energy in the box. [We used the substitution $v = ct - d$ in the first line.]

PINGBACKS

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