GAUGE TRANSFORMATIONS IN ELECTRODYNAMICS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.5.

The potentials V and $\mathbf{A}$ that give rise to a particular configuration of fields $\mathbf{E}$ and $\mathbf{B}$ are not unique. For example, for a point charge at the origin, the usual potentials would be

$$\begin{aligned} V &= \frac{q}{4\pi\epsilon_0 r} && (1) \\ \mathbf{A} &= 0 && (2) \end{aligned}$$

but as we've seen, the potentials

$$\begin{aligned} V(\mathbf{r},t) &= 0 && (3) \\ \mathbf{A}(\mathbf{r},t) &= -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{\mathbf{r}} && (4) \end{aligned}$$

give rise to the same fields. The question arises: what is the most general set of potentials that give a particular configuration of fields? That is, what changes can we make to V and $\mathbf{A}$ without changing $\mathbf{E}$ and $\mathbf{B}$?

The fields can be calculated from the potentials via the equations

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A} && (5) \\ \mathbf{E} &= -\nabla V - \frac{\partial\mathbf{A}}{\partial t} && (6) \end{aligned}$$

We can't multiply V and $\mathbf{A}$ by anything and leave the fields unchanged, but we might be able to add something. Suppose we add a vector function $\alpha(\mathbf{r},t)$ to $\mathbf{A}$ and a scalar function $\beta(\mathbf{r},t)$ to V:

$$\begin{aligned} \mathbf{A}' &= \mathbf{A}+\alpha && (7) \\ V' &= V+\beta && (8) \end{aligned}$$

To get the same magnetic field, we must have

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} && (9)\\
&= \nabla\times\mathbf{A}' && (10)\\
&= \nabla\times(\mathbf{A}+\alpha) && (11)\\
\nabla\times\alpha &= 0 && (12)
\end{aligned}$$

Since its curl is zero, we can write α as the gradient of a scalar field:

$$\alpha(\mathbf{r},t) = \nabla\lambda'(\mathbf{r},t) \tag{13}$$

We also have to get the same electric field $\mathbf{E}$ so

$$\begin{aligned}
\mathbf{E} &= -\nabla V - \frac{\partial\mathbf{A}}{\partial t} && (14)\\
&= -\nabla(V+\beta) - \frac{\partial(\mathbf{A}+\alpha)}{\partial t} && (15)\\
\nabla\beta + \frac{\partial\alpha}{\partial t} &= 0 && (16)
\end{aligned}$$

Combining these results, we get

$$\nabla\left(\beta + \frac{\partial\lambda'}{\partial t}\right) = 0 \tag{17}$$

The term in parentheses must not depend on position (since its gradient is identically zero everywhere), but it might depend on time. Define this term to be $k(t)$; then

$$\beta = k(t) - \frac{\partial\lambda'}{\partial t} \tag{18}$$

If we define

$$\lambda \equiv \lambda' - \int_0^t k(t')\,dt' \tag{19}$$

then

$$\beta = -\frac{\partial\lambda}{\partial t} \tag{20}$$

Since λ and λ' differ only by a function of time (and not of space), their gradients are equal, so we can use λ in place of λ' in 13 without changing α. Therefore we can change the potentials as follows, without changing the fields they produce:

$$\mathbf{A}' = \mathbf{A}+\nabla\lambda \tag{21}$$

$$V' = V-\frac{\partial\lambda}{\partial t} \tag{22}$$

where $\lambda=\lambda(\mathbf{r},t)$ is an arbitrary scalar field. Such a change to the potentials is called a *gauge transformation*.

Example. Earlier, we saw the unusual potentials 3 and 4 for a point charge at the origin. We can transform them using the gauge function

$$\lambda=-\frac{qt}{4\pi\epsilon_0 r} \tag{23}$$

We get

$$\nabla\lambda = \frac{qt}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{24}$$

$$\frac{\partial\lambda}{\partial t} = -\frac{q}{4\pi\epsilon_0 r} \tag{25}$$

so the new potentials are

$$V = \frac{q}{4\pi\epsilon_0 r} \tag{26}$$

$$\mathbf{A} = 0 \tag{27}$$

which is the more usual set of potentials 1 and 2 for a point charge.

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