RETARDED POTENTIALS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Section 10.2.1. Problem 10.8.

In electrostatics and magnetostatics, charge distributions and currents were all constant in time. When they vary, we need to take into account the finite speed of light in calculating potentials and fields. If we want the fields at some point P then, if the charge or current changes at some point Q a distance d from P, an observer at P won't know about the change until the signal from Q reaches him, which in vacuum takes a time d/c. To take account of this, the potentials at position $\mathbf{r}$ and time t in a dynamic system are taken to be

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{2}$$

where

$$t_r \equiv t - \frac{d}{c} \tag{3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{6}$$

That is, each potential is the sum over all locations where there is charge or current, and each location $\mathbf{r}'$ is sampled at the time t_r in the past which is the time a light signal would have left $\mathbf{r}'$ to arrive at $\mathbf{r}$ at time t. These potentials are called *retarded potentials*, since they depend on the situation at various times in the past to get the fields at the present time.

Griffiths shows in his section 10.2.1 that these potentials (well V anyway; the argument for $\mathbf{A}$ is similar) satisfy the wave equations in the Lorenz gauge

$$\square^2 V = -\frac{\rho}{\epsilon_0} \tag{7}$$

$$\square^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{8}$$

We also need to show that the potentials satisfy the Lorenz gauge condition

$$\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{9}$$

Starting from 2 we need to find

$$\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\mathbf{J}\left(\mathbf{r}',t_r\right)}{d}\right)d^3\mathbf{r}' \tag{10}$$

Note that the ∇ is a derivative with respect to $\mathbf{r}$ (the observer's position) and not $\mathbf{r}'$ (the source positions and variable of integration), and that both t_r and d depend on both $\mathbf{r}$ and $\mathbf{r}'$. We begin by writing

$$\nabla\cdot\left(\frac{\mathbf{J}}{d}\right) = \frac{1}{d}\nabla\cdot\mathbf{J} + \mathbf{J}\cdot\nabla\left(\frac{1}{d}\right) \tag{11}$$

We can also use the derivative with respect to $\mathbf{r}'$:

$$\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) = \frac{1}{d}\nabla'\cdot\mathbf{J} + \mathbf{J}\cdot\nabla'\left(\frac{1}{d}\right) \tag{12}$$

We have (you can work this out by using 5 if you don't believe me):

$$\nabla\left(\frac{1}{d}\right) = -\frac{\hat{\mathbf{d}}}{d^2} = -\nabla'\left(\frac{1}{d}\right) \tag{13}$$

Therefore

$$\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) = \frac{1}{d}\nabla'\cdot\mathbf{J} - \mathbf{J}\cdot\nabla\left(\frac{1}{d}\right) \tag{14}$$

$$\mathbf{J}\cdot\nabla\left(\frac{1}{d}\right) = \frac{1}{d}\nabla'\cdot\mathbf{J} - \nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) \tag{15}$$

Inserting this into 11 we get

$$\nabla\cdot\left(\frac{\mathbf{J}}{d}\right)=\frac{1}{d}\nabla\cdot\mathbf{J}+\frac{1}{d}\nabla'\cdot\mathbf{J}-\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) \tag{16}$$

Now we need to work out the two divergences $\nabla\cdot\mathbf{J}$ and $\nabla\cdot\mathbf{J}'$. To do this, we need to remember that $\mathbf{J}=\mathbf{J}(\mathbf{r}',t_r)$, so it depends on $\mathbf{r}$ only via t_r but it depends on $\mathbf{r}'$ both explicitly through its first argument and implicitly through t_r. Using the chain rule, we get for the contribution from x

$$\begin{aligned}(\nabla\cdot\mathbf{J})_x &= \frac{\partial\mathbf{J}_x}{\partial t_r}\frac{\partial t_r}{\partial x} && (17)\\ &= -\frac{1}{c}\dot{\mathbf{J}}_x\frac{\partial d}{\partial x} && (18)\\ &= -\frac{1}{c}\dot{\mathbf{J}}_x(\nabla d)_x && (19)\end{aligned}$$

where the dot over the $\mathbf{J}$ is a derivative with respect to t, which is the same as a derivative with respect to t_r since $t_r=t-d/c$ and d doesn't depend on time.

The other two coordinates give similar results and we get

$$\nabla\cdot\mathbf{J}=-\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla d \tag{20}$$

For the other divergence, things are a bit trickier since $\mathbf{J}$ depends explicitly on $\mathbf{r}'$. Here we used the extended chain rule

$$\frac{\partial g(x,f(x))}{\partial x}=\frac{\partial g}{\partial x}+\frac{\partial g}{\partial f}\frac{\partial f}{\partial x} \tag{21}$$

Therefore

$$\nabla'\cdot\mathbf{J}=\nabla'_{\mathbf{r}'}\cdot\mathbf{J}-\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla' d \tag{22}$$

where we've used the specialized notation $\nabla'_{\mathbf{r}'}$ to indicate the divergence with respect to the *explicit* $\mathbf{r}'$ dependence in $\mathbf{J}$. From Maxwell's fourth equation

$$\nabla'_{\mathbf{r}'}\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{23}$$

where the fields and currents depend on $\mathbf{r}'$, we can take the explicit divergence to get

$$\nabla'_{\mathbf{r}'}\cdot\left(\nabla'_{\mathbf{r}'}\times\mathbf{B}\right)=\mu_0\nabla'_{\mathbf{r}'}\cdot\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\nabla'_{\mathbf{r}'}\cdot\mathbf{E}}{\partial t} \tag{24}$$

The divergence of a curl is always zero, so we get

$$\begin{aligned}\nabla'_{\mathbf{r}'}\cdot\mathbf{J} &= -\varepsilon_0\frac{\partial\nabla'_{\mathbf{r}'}\cdot\mathbf{E}}{\partial t} &(25)\\ &= -\dot{\rho} &(26)\end{aligned}$$

where we've used Maxwell's first equation

$$\nabla'_{\mathbf{r}'}\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0} \tag{27}$$

Plugging this into 22 we get

$$\begin{aligned}\nabla'\cdot\mathbf{J} &= -\dot{\rho}-\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla' d &(28)\\ &= -\dot{\rho}+\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla d &(29)\end{aligned}$$

since from 5

$$\nabla' d=-\nabla d \tag{30}$$

Putting 20 and 29 into 16 we get

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{J}}{d}\right) &= \frac{1}{d}\left[-\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla d-\dot{\rho}+\frac{1}{c}\dot{\mathbf{J}}\cdot\nabla d\right]-\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) &(31)\\ &= -\frac{\dot{\rho}}{d}-\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right) &(32)\end{aligned}$$

Finally, from 2 we have

$$\nabla\cdot\mathbf{A}\left(\mathbf{r},t\right)=\frac{\mu_0}{4\pi}\int\left[-\frac{\dot{\rho}}{d}-\nabla'\cdot\left(\frac{\mathbf{J}}{d}\right)\right]d^3\mathbf{r}' \tag{33}$$

Using the divergence theorem, the second term can be converted to a surface integral at infinity where (presumably) the current $\mathbf{J}$ is zero, so this term vanishes. Using 1 we then get

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\int\frac{\dot{\rho}}{d}d^3\mathbf{r}' \tag{34}$$

$$= -\mu_0\varepsilon_0\frac{\partial V}{\partial t} \tag{35}$$

which is the Lorenz gauge condition, as required.

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