RETARDED POTENTIALS IN AN INFINITE STRAIGHT WIRE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.9.

Here are a couple of examples of calculating the retarded potential. In both examples, we have an infinite straight wire that carries a current. At time $t=0$ a current $I(t)$ is switched on, and we want to find the electric and magnetic fields after that time. We'll assume the wire is electrically neutral so there is no free charge density and the scalar potential is thus $V=0$.

Example 1. The current is linearly increasing:

$$I(t) = kt \tag{1}$$

For a linear current the vector potential is (assuming the current flows in the z direction):

$$\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int\frac{I(\mathbf{r}',t_r)}{d}dz \tag{2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{3}$$

and

$$d = \sqrt{r^2+z^2} \tag{4}$$

where we're using cylindrical coordinates, so r is the perpendicular distance from the wire and z is the distance along the wire.

At time t and distance r, only those parts of the wire where $d<ct$ will contribute to the potential, so the integral's limits are $z=\pm\sqrt{(ct)^2-r^2}$ and we get

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 k}{4\pi}\int_{-\sqrt{(ct)^2-r^2}}^{\sqrt{(ct)^2-r^2}}\frac{t-\sqrt{r^2+z^2}/c}{\sqrt{r^2+z^2}}dz \tag{5}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{4\pi c}2\int_0^{\sqrt{(ct)^2-r^2}}\left[\frac{ct}{\sqrt{r^2+z^2}}-1\right]dz \tag{6}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi c}\left[ct\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)-\sqrt{(ct)^2-r^2}\right] \tag{7}$$

From this we can find **E** and **B** using the equations

$$\mathbf{E} = -\nabla V-\frac{\partial \mathbf{A}}{\partial t} \tag{8}$$

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{9}$$

The derivatives are messy and best done with Maple. The results are

$$\mathbf{E} = -\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{\mathbf{z}} \tag{10}$$

$$\mathbf{B} = \frac{\mu_0 k\left(\sqrt{(ct)^2-r^2}ct+(ct)^2-r^2\right)}{2\pi rc\left(ct+\sqrt{(ct)^2-r^2}\right)}\hat{\theta} \tag{11}$$

$$= \frac{\mu_0 k\sqrt{(ct)^2-r^2}}{2\pi rc}\hat{\theta} \tag{12}$$

Example 2. This time the current is an impulse at $t=0$ given by

$$I(t) = q_0\delta(t) \tag{13}$$

The argument above still applies; only the integrand is different. We get

$$\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int\frac{I(\mathbf{r}',t_r)}{d}dz \tag{14}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 q_0}{2\pi}\int_0^{\sqrt{(ct)^2-r^2}}\frac{\delta\left(t-\sqrt{r^2+z^2}/c\right)}{\sqrt{r^2+z^2}}dz \tag{15}$$

We can transform the integral using the substitution

$$u = t - \frac{\sqrt{r^2+z^2}}{c} \tag{16}$$

$$\sqrt{r^2+z^2} = ct - cu \tag{17}$$

$$z = \sqrt{(ct-cu)^2 - r^2} \tag{18}$$

$$dz = -\frac{c\,(ct-cu)}{\sqrt{(ct-cu)^2-r^2}}du \tag{19}$$

The limits transform as

$$z = 0 \quad \rightarrow \quad u = t - \frac{r}{c} \tag{20}$$

$$z = \sqrt{(ct)^2 - r^2} \quad \rightarrow \quad u = 0 \tag{21}$$

The integral becomes

$$\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}\frac{\mu_0 q_0}{2\pi}\int_0^{t-\frac{r}{c}} \frac{c\,(ct-cu)\,\delta(u)}{(ct-cu)\sqrt{(ct-cu)^2-r^2}}du \tag{22}$$

$$= \hat{\mathbf{z}}\frac{c\mu_0 q_0}{2\pi\sqrt{(ct)^2-r^2}} \tag{23}$$

The corresponding fields are

$$\mathbf{E} = \frac{\mu_0 q_0 c^3 t}{2\pi\,(c^2t^2-r^2)^{3/2}}\hat{\mathbf{z}} \tag{24}$$

$$\mathbf{B} = -\frac{\mu_0 q_0 c r}{2\pi\,(c^2t^2-r^2)^{3/2}}\hat{\theta} \tag{25}$$