

JEFIMENKO'S EQUATION FOR TIME-DEPENDENT ELECTRIC FIELD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.11.

Using the retarded potentials, we can find a time-dependent expression for the electric field \mathbf{E} . The potentials are

$$(1) \quad V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{d} d^3\mathbf{r}'$$

$$(2) \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{d} d^3\mathbf{r}'$$

where

$$(3) \quad t_r \equiv t - \frac{d}{c}$$

and

$$(4) \quad d \equiv |\mathbf{r} - \mathbf{r}'|$$

$$(5) \quad = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$(6) \quad \hat{\mathbf{d}} = \frac{\mathbf{r} - \mathbf{r}'}{d}$$

The expression for \mathbf{E} is given by

$$(7) \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

The derivatives are complicated by the fact that the integrands in 1 and 2 depend on \mathbf{r} both via t_r and d . We get (note that ∇ indicates derivatives with respect to components of \mathbf{r} only, not \mathbf{r}'):

$$(8) \quad \nabla V = \frac{1}{4\pi\epsilon_0} \int \nabla \left(\frac{\rho(\mathbf{r}', t_r)}{d} \right) d^3\mathbf{r}'$$

$$(9) \quad = \frac{1}{4\pi\epsilon_0} \int \rho \nabla \left(\frac{1}{d} \right) + \frac{1}{d} \nabla \rho d^3\mathbf{r}'$$

Using the chain rule

$$(10) \quad \nabla \rho = \frac{\partial \rho}{\partial t_r} \nabla t_r$$

$$(11) \quad = -\frac{1}{c} \frac{\partial \rho}{\partial t} \nabla d$$

$$(12) \quad = -\frac{\dot{\rho}}{c} \nabla d$$

since $\frac{\partial \rho}{\partial t_r} = \frac{\partial \rho}{\partial t}$ because of 3. By direct calculation we have

$$(13) \quad \nabla d = \hat{\mathbf{d}}$$

$$(14) \quad \nabla \left(\frac{1}{d} \right) = -\frac{\hat{\mathbf{d}}}{d^2}$$

Plugging everything into 9 we get

$$(15) \quad \nabla V(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho}{d^2} + \frac{\dot{\rho}}{cd} \right) \hat{\mathbf{d}} d^3\mathbf{r}'$$

The second term in 7 is just

$$(16) \quad \frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{d} d^3\mathbf{r}'$$

so the time-dependent field is (using $\mu_0\epsilon_0 = 1/c^2$):

$$(17) \quad \mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho}{d^2} + \frac{\dot{\rho}}{cd} \right) \hat{\mathbf{d}} d^3\mathbf{r}' - \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{d} d^3\mathbf{r}'$$

$$(18) \quad = \frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho(\mathbf{r}', t_r)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cd} \hat{\mathbf{d}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2 d} \right) d^3\mathbf{r}'$$

This is Jefimenko's equation for the electric field. In the static case, all time derivatives are zero and there is no dependence on t_r so we get

$$(19) \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{d^2} \hat{\mathbf{d}} d^3\mathbf{r}'$$

which is just Coulomb's law from electrostatics.

A special case is that of constant current but varying charge. In that case

$$(20) \quad \rho(\mathbf{r}, t) = \dot{\rho}(\mathbf{r}, 0)t + \rho(\mathbf{r}, 0)$$

where

$$(21) \quad \dot{\rho}(\mathbf{r}, 0) = -\nabla \cdot \mathbf{J}(\mathbf{r})$$

In this case, the integrand of 18 is

$$(22) \quad \frac{\rho(\mathbf{r}', t_r)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cd} \hat{\mathbf{d}} - \frac{\mathbf{J}(\mathbf{r}', t_r)}{c^2 d} = \frac{\dot{\rho}(\mathbf{r}', 0)(t - \frac{d}{c}) + \rho(\mathbf{r}', 0)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', 0)}{cd} \hat{\mathbf{d}}$$

$$(23) \quad = \frac{\dot{\rho}(\mathbf{r}', 0)t + \rho(\mathbf{r}', 0)}{d^2} \hat{\mathbf{d}}$$

$$(24) \quad = \frac{\rho(\mathbf{r}', t)}{d} \hat{\mathbf{d}}$$

so the field is

$$(25) \quad \mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{d^2} \hat{\mathbf{d}} d^3\mathbf{r}'$$

That is, Coulomb's law is valid with the charge density evaluated at the current (non-retarded) time.

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