JEFIMENKO'S EQUATION FOR TIME-DEPENDENT ELECTRIC FIELD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.11.

Using the retarded potentials, we can find a time-dependent expression for the electric field $\mathbf{E}$. The potentials are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{6}$$

The expression for $\mathbf{E}$ is given by

$$\mathbf{E} = -\nabla V-\frac{\partial\mathbf{A}}{\partial t} \tag{7}$$

The derivatives are complicated by the fact that the integrands in 1 and 2 depend on $\mathbf{r}$ both via t_r and d. We get (note that ∇ indicates derivatives with respect to components of $\mathbf{r}$ only, not $\mathbf{r}'$):

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\nabla\left(\frac{\rho(\mathbf{r}',t_r)}{d}\right)d^3\mathbf{r}' \tag{8}$$

$$= \frac{1}{4\pi\epsilon_0}\int\rho\nabla\left(\frac{1}{d}\right)+\frac{1}{d}\nabla\rho d^3\mathbf{r}' \tag{9}$$

Using the chain rule

$$\nabla\rho = \frac{\partial\rho}{\partial t_r}\nabla t_r \tag{10}$$
$$= -\frac{1}{c}\frac{\partial\rho}{\partial t}\nabla d \tag{11}$$
$$= -\frac{\dot{\rho}}{c}\nabla d \tag{12}$$

since $\frac{\partial\rho}{\partial t_r} = \frac{\partial\rho}{\partial t}$ because of 3. By direct calculation we have

$$\nabla d = \hat{\mathbf{d}} \tag{13}$$
$$\nabla\left(\frac{1}{d}\right) = -\frac{\hat{\mathbf{d}}}{d^2} \tag{14}$$

Plugging everything into 9 we get

$$\nabla V(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho}{d^2}+\frac{\dot{\rho}}{cd}\right)\hat{\mathbf{d}}d^3\mathbf{r}' \tag{15}$$

The second term in 7 is just

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{16}$$

so the time-dependent field is (using $\mu_0\epsilon_0 = 1/c^2$):

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho}{d^2}+\frac{\dot{\rho}}{cd}\right)\hat{\mathbf{d}}d^3\mathbf{r}' - \frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{17}$$
$$= \frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho(\mathbf{r}',t_r)}{d^2}\hat{\mathbf{d}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{cd}\hat{\mathbf{d}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2d}\right)d^3\mathbf{r}' \tag{18}$$

This is Jefimenko's equation for the electric field. In the static case, all time derivatives are zero and there is no dependence on t_r so we get

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{d^2}\hat{\mathbf{d}}d^3\mathbf{r}' \tag{19}$$

which is just Coulomb's law from electrostatics.

A special case is that of constant current but varying charge. In that case

$$\rho(\mathbf{r},t) = \dot{\rho}(\mathbf{r},0)\,t+\rho(\mathbf{r},0) \tag{20}$$

where

$$\dot{\rho}(\mathbf{r},0) = -\nabla\cdot\mathbf{J}(\mathbf{r}) \tag{21}$$

In this case, the integrand of 18 is

$$\frac{\rho(\mathbf{r}',t_r)}{d^2}\hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}',t_r)}{cd}\hat{\mathbf{d}} - \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2 d} = \frac{\dot{\rho}(\mathbf{r}',0)\left(t-\frac{d}{c}\right)+\rho(\mathbf{r}',0)}{d^2}\hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}',0)}{cd}\hat{\mathbf{d}} \tag{22}$$

$$= \frac{\dot{\rho}(\mathbf{r}',0)\,t+\rho(\mathbf{r}',0)}{d^2}\hat{\mathbf{d}} \tag{23}$$

$$= \frac{\rho(\mathbf{r}',t)}{d}\hat{\mathbf{d}} \tag{24}$$

so the field is

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t)}{d^2}\hat{\mathbf{d}}d^3\mathbf{r}' \tag{25}$$

That is, Coulomb's law is valid with the charge density evaluated at the current (non-retarded) time.

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