JEFIMENKO'S EQUATION FOR TIME-DEPENDENT ELECTRIC FIELD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.11.

Using the retarded potentials, we can find a time-dependent expression for the electric field $\mathbf{E}$. The potentials are

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{0.3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{0.4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{0.5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{0.6}$$

The expression for $\mathbf{E}$ is given by

$$\mathbf{E} = -\nabla V-\frac{\partial\mathbf{A}}{\partial t} \tag{0.7}$$

The derivatives are complicated by the fact that the integrands in 0.1 and 0.2 depend on $\mathbf{r}$ both via t_r and d. We get (note that ∇ indicates derivatives with respect to components of $\mathbf{r}$ only, not $\mathbf{r}'$):

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int\nabla\left(\frac{\rho\left(\mathbf{r}',t_r\right)}{d}\right)d^3\mathbf{r}' \tag{0.8}$$

$$= \frac{1}{4\pi\varepsilon_0}\int\rho\nabla\left(\frac{1}{d}\right)+\frac{1}{d}\nabla\rho d^3\mathbf{r}' \tag{0.9}$$

Using the chain rule

$$\nabla\rho = \frac{\partial\rho}{\partial t_r}\nabla t_r \tag{0.10}$$

$$= -\frac{1}{c}\frac{\partial\rho}{\partial t}\nabla d \tag{0.11}$$

$$= -\frac{\dot{\rho}}{c}\nabla d \tag{0.12}$$

since $\frac{\partial\rho}{\partial t_r}=\frac{\partial\rho}{\partial t}$ because of 0.3. By direct calculation we have

$$\nabla d = \hat{\mathbf{d}} \tag{0.13}$$

$$\nabla\left(\frac{1}{d}\right) = -\frac{\hat{\mathbf{d}}}{d^2} \tag{0.14}$$

Plugging everything into 0.9 we get

$$\nabla V\left(\mathbf{r},t\right)=-\frac{1}{4\pi\varepsilon_0}\int\left(\frac{\rho}{d^2}+\frac{\dot{\rho}}{cd}\right)\hat{\mathbf{d}}d^3\mathbf{r}' \tag{0.15}$$

The second term in 0.7 is just

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}\left(\mathbf{r}',t_r\right)}{d}d^3\mathbf{r}' \tag{0.16}$$

so the time-dependent field is (using $\mu_0\varepsilon_0=1/c^2$):

$$\mathbf{E}\left(\mathbf{r},t\right)=\frac{1}{4\pi\varepsilon_0}\int\left(\frac{\rho}{d^2}+\frac{\dot{\rho}}{cd}\right)\hat{\mathbf{d}}d^3\mathbf{r}'-\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}\left(\mathbf{r}',t_r\right)}{d}d^3\mathbf{r}' \tag{0.17}$$

$$=\frac{1}{4\pi\varepsilon_0}\int\left(\frac{\rho\left(\mathbf{r}',t_r\right)}{d^2}\hat{\mathbf{d}}+\frac{\dot{\rho}\left(\mathbf{r}',t_r\right)}{cd}\hat{\mathbf{d}}-\frac{\dot{\mathbf{J}}\left(\mathbf{r}',t_r\right)}{c^2d}\right)d^3\mathbf{r}' \tag{0.18}$$

This is Jefimenko's equation for the electric field. In the static case, all time derivatives are zero and there is no dependence on t_r so we get

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{d^2}\hat{\mathbf{d}}d^3\mathbf{r}' \tag{0.19}$$

which is just Coulomb's law from electrostatics.

A special case is that of constant current but varying charge. In that case

$$\rho(\mathbf{r},t)=\dot{\rho}(\mathbf{r},0)t+\rho(\mathbf{r},0) \tag{0.20}$$

where

$$\dot{\rho}(\mathbf{r},0)=-\nabla\cdot\mathbf{J}(\mathbf{r}) \tag{0.21}$$

In this case, the integrand of 0.18 is

$$\frac{\rho(\mathbf{r}',t_r)}{d^2}\hat{\mathbf{d}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{cd}\hat{\mathbf{d}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2d}=\frac{\dot{\rho}(\mathbf{r}',0)\left(t-\frac{d}{c}\right)+\rho(\mathbf{r}',0)}{d^2}\hat{\mathbf{d}}+\frac{\dot{\rho}(\mathbf{r}',0)}{cd}\hat{\mathbf{d}} \tag{0.22}$$

$$=\frac{\dot{\rho}(\mathbf{r}',0)t+\rho(\mathbf{r}',0)}{d^2}\hat{\mathbf{d}} \tag{0.23}$$

$$=\frac{\rho(\mathbf{r}',t)}{d}\hat{\mathbf{d}} \tag{0.24}$$

so the field is

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t)}{d^2}\hat{\mathbf{d}}d^3\mathbf{r}' \tag{0.25}$$

That is, Coulomb's law is valid with the charge density evaluated at the current (non-retarded) time.

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