

JEFIMENKO'S EQUATION FOR TIME-DEPENDENT ELECTRIC FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.11.

Using the retarded potentials, we can find a time-dependent expression for the electric field \mathbf{E} . The potentials are

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{d} d^3\mathbf{r}' \quad (1)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{d} d^3\mathbf{r}' \quad (2)$$

where

$$t_r \equiv t - \frac{d}{c} \quad (3)$$

and

$$d \equiv |\mathbf{r} - \mathbf{r}'| \quad (4)$$

$$= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad (5)$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r} - \mathbf{r}'}{d} \quad (6)$$

The expression for \mathbf{E} is given by

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (7)$$

The derivatives are complicated by the fact that the integrands in 1 and 2 depend on \mathbf{r} both via t_r and d . We get (note that ∇ indicates derivatives with respect to components of \mathbf{r} only, not \mathbf{r}'):

$$\nabla V = \frac{1}{4\pi\epsilon_0} \int \nabla \left(\frac{\rho(\mathbf{r}', t_r)}{d} \right) d^3\mathbf{r}' \quad (8)$$

$$= \frac{1}{4\pi\epsilon_0} \int \rho \nabla \left(\frac{1}{d} \right) + \frac{1}{d} \nabla \rho d^3\mathbf{r}' \quad (9)$$

Using the chain rule

$$\nabla \rho = \frac{\partial \rho}{\partial t_r} \nabla t_r \quad (10)$$

$$= -\frac{1}{c} \frac{\partial \rho}{\partial t} \nabla d \quad (11)$$

$$= -\frac{\dot{\rho}}{c} \nabla d \quad (12)$$

since $\frac{\partial \rho}{\partial t_r} = \frac{\partial \rho}{\partial t}$ because of 3. By direct calculation we have

$$\nabla d = \hat{\mathbf{d}} \quad (13)$$

$$\nabla \left(\frac{1}{d} \right) = -\frac{\hat{\mathbf{d}}}{d^2} \quad (14)$$

Plugging everything into 9 we get

$$\nabla V(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho}{d^2} + \frac{\dot{\rho}}{cd} \right) \hat{\mathbf{d}} d^3 \mathbf{r}' \quad (15)$$

The second term in 7 is just

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{d} d^3 \mathbf{r}' \quad (16)$$

so the time-dependent field is (using $\mu_0\epsilon_0 = 1/c^2$):

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho}{d^2} + \frac{\dot{\rho}}{cd} \right) \hat{\mathbf{d}} d^3 \mathbf{r}' - \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{d} d^3 \mathbf{r}' \quad (17)$$

$$= \frac{1}{4\pi\epsilon_0} \int \left(\frac{\rho(\mathbf{r}', t_r)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cd} \hat{\mathbf{d}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2 d} \right) d^3 \mathbf{r}' \quad (18)$$

This is Jefimenko's equation for the electric field. In the static case, all time derivatives are zero and there is no dependence on t_r so we get

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{d^2} \hat{\mathbf{d}} d^3 \mathbf{r}' \quad (19)$$

which is just Coulomb's law from electrostatics.

A special case is that of constant current but varying charge. In that case

$$\rho(\mathbf{r}, t) = \dot{\rho}(\mathbf{r}, 0)t + \rho(\mathbf{r}, 0) \quad (20)$$

where

$$\dot{\rho}(\mathbf{r}, 0) = -\nabla \cdot \mathbf{J}(\mathbf{r}) \quad (21)$$

In this case, the integrand of 18 is

$$\frac{\rho(\mathbf{r}', t_r)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cd} \hat{\mathbf{d}} - \frac{\mathbf{J}(\mathbf{r}', t_r)}{c^2 d} = \frac{\dot{\rho}(\mathbf{r}', 0) \left(t - \frac{d}{c}\right) + \rho(\mathbf{r}', 0)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(\mathbf{r}', 0)}{cd} \hat{\mathbf{d}} \quad (22)$$

$$= \frac{\dot{\rho}(\mathbf{r}', 0)t + \rho(\mathbf{r}', 0)}{d^2} \hat{\mathbf{d}} \quad (23)$$

$$= \frac{\rho(\mathbf{r}', t)}{d} \hat{\mathbf{d}} \quad (24)$$

so the field is

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{d^2} \hat{\mathbf{d}} d^3\mathbf{r}' \quad (25)$$

That is, Coulomb's law is valid with the charge density evaluated at the current (non-retarded) time.

PINGBACKS

Pingback: Jefimenko's equation for time-dependent magnetic field

Pingback: Relations among charge, current, potentials and fields

Pingback: Electric dipole time-varying potentials; Lorenz gauge condition