JEFIMENKO'S EQUATION FOR TIME-DEPENDENT MAGNETIC FIELD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.12.

Using the retarded potentials, we can find a time-dependent expression for the magnetic field $\mathbf{B}$ to complement the equation we got earlier for the electric field. The potentials are

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{0.3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{0.4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{0.5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{0.6}$$

The magnetic field is

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{0.7}$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{(\nabla\times\mathbf{J}(\mathbf{r}',t_r))}{d}-\mathbf{J}\left(\mathbf{r}',t_r\right)\times\nabla\left(\frac{1}{d}\right)\right]d^3\mathbf{r}' \tag{0.8}$$

where we've used the identity

$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A})-\mathbf{A}\times\nabla f \tag{0.9}$$

We have

$$\nabla\left(\frac{1}{d}\right)=-\frac{\hat{\mathbf{d}}}{d^{2}} \tag{0.10}$$

To get $\nabla\times\mathbf{J}$, it's easiest to look at a single component of the curl, say the x component:

$$(\nabla\times\mathbf{J})_{x}=\frac{\partial J_{z}}{\partial y}-\frac{\partial J_{y}}{\partial z} \tag{0.11}$$

Using the chain rule

$$\begin{aligned}\frac{\partial J_{z}}{\partial y} &= \frac{\partial J_{z}}{\partial t_{r}}\frac{\partial t_{r}}{\partial y} && (0.12)\\ &= -\frac{1}{c}\frac{\partial J_{z}}{\partial t}\frac{\partial d}{\partial y} && (0.13)\\ &= -\frac{1}{c}\dot{J}_{z}(\nabla d)_{y} && (0.14)\end{aligned}$$

Therefore

$$\begin{aligned}(\nabla\times\mathbf{J})_{x} &= -\frac{1}{c}\left(\dot{J}_{z}(\nabla d)_{y}-\dot{J}_{y}(\nabla d)_{z}\right) && (0.15)\\ &= \frac{1}{c}\left(\dot{\mathbf{J}}\times\nabla d\right)_{x} && (0.16)\end{aligned}$$

The other two components work out the same way so we have

$$\begin{aligned}\nabla\times\mathbf{J} &= \frac{1}{c}\dot{\mathbf{J}}\times\nabla d && (0.17)\\ &= \frac{1}{c}\dot{\mathbf{J}}\times\hat{\mathbf{d}} && (0.18)\end{aligned}$$

Putting this back into 0.8 we get

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_{0}}{4\pi}\int\left[\frac{\dot{\mathbf{J}}(\mathbf{r}',t_{r})\times\hat{\mathbf{d}}}{cd}+\mathbf{J}\left(\mathbf{r}',t_{r}\right)\times\frac{\hat{\mathbf{d}}}{d^{2}}\right]d^{3}\mathbf{r}' \tag{0.19}$$

This is Jefimenko's equation for the magnetic field. For steady currents, $\dot{\mathbf{J}}=0$ and the dependence on t_r disappears, so we're left with the Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}\left(\mathbf{r}'\right) \times \frac{\hat{\mathbf{d}}}{d^2} d^3\mathbf{r}' \tag{0.20}$$

In the special case where the current changes slowly enough that we can approximate it by a first-order Taylor series:

$$\mathbf{J}(\mathbf{r},t_r) = \mathbf{J}(\mathbf{r},t) + (t_r - t)\dot{\mathbf{J}}(\mathbf{r},t) \tag{0.21}$$

$$= \mathbf{J}(\mathbf{r},t) - \frac{d}{c}\dot{\mathbf{J}}(\mathbf{r},t) \tag{0.22}$$

$$\frac{\partial \mathbf{J}}{\partial t_r} = \dot{\mathbf{J}}(\mathbf{r},t) \tag{0.23}$$

we get from 0.19

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \left[\frac{\dot{\mathbf{J}}(\mathbf{r}',t)}{cd} + \frac{1}{d^2}\left(\mathbf{J}\left(\mathbf{r}',t\right) - \frac{d}{c}\dot{\mathbf{J}}\left(\mathbf{r}',t\right)\right)\right] \times \hat{\mathbf{d}} d^3\mathbf{r}' \tag{0.24}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t) \times \hat{\mathbf{d}}}{d^2} d^3\mathbf{r}' \tag{0.25}$$

That is, for slowly varying currents, to first order, Jefimenko's equation gives the Biot-Savart law where all currents are evaluated at the current time. We're essentially assuming that the travel time for signals from all locations of the current to the observation point is zero. This was the quasistatic approximation.

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