LIÉNARD-WIECHERT POTENTIALS FOR A MOVING POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.13.

We now look at the retarded potentials for a moving point charge q. The potentials are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{6}$$

The charge density of a point charge is represented by a delta function in space, so if the charge's trajectory is given by $\mathbf{w}(t')$ then

$$\rho\left(\mathbf{r}',t'\right) = q\delta^3\left(\mathbf{r}'-\mathbf{w}\left(t'\right)\right) \tag{7}$$

To work out V, we need the charge density at the retarded time t_r, which we can write as the integral over time of the charge density multiplied by another delta function:

$$\rho\left(\mathbf{r}',t_r\right) = q\delta^3\left(\mathbf{r}'-\mathbf{w}\left(t'\right)\right)\int\delta\left(t'-t_r\right)dt' \tag{8}$$

We need to keep straight the different times we're using here. The time t is the observation time, t' is the integration variable and t_r is the retarded time, which is the time at which the signal that we are receiving at time t left the moving charge, which is

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{9}$$

The potential can now be written as an integral over both time and space:

$$\begin{aligned} V(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{\delta^3(\mathbf{r}'-\mathbf{w}(t'))}{|\mathbf{r}-\mathbf{r}'|}\int dt'\delta\left(t'-t_r\right) && (10)\\ &= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{\delta^3(\mathbf{r}'-\mathbf{w}(t'))}{|\mathbf{r}-\mathbf{r}'|}\int dt'\delta\left(t'-\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right) && (11)\end{aligned}$$

We can do the spatial integration which sets $\mathbf{r}' = \mathbf{w}(t')$

$$\frac{4\pi\epsilon_0}{q}V(\mathbf{r},t) = \int dt'\frac{1}{|\mathbf{r}-\mathbf{w}(t')|}\delta\left(t'-\left(t-\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right)\right) \tag{12}$$

The trick now is to transform the argument of the delta function so we can do the integral. To do this, we need to work out $\delta(f(x))$ for some function $f(x)$. To work this out, we use the substitution

$$\begin{aligned} u &= f(x) && (13)\\ du &= f'(x)\,dx && (14)\end{aligned}$$

so we get

$$\begin{aligned} \int\delta(f(x))\,dx &= \int\frac{\delta(u)}{|f'(x)|}du && (15)\\ &= \frac{1}{|f'(x(0))|} && (16)\end{aligned}$$

where we need to solve for x as a function of u from 13 and then find $x(u=0)$.

For our problem, we have

$$\begin{aligned} f\left(t'\right) &= t'-\left(t-\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right) && (17)\\ \frac{df}{dt'} &= 1+\frac{1}{c}\frac{d}{dt'}\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right| && (18)\end{aligned}$$

Let's select our coordinate axes so that, at time t', $\mathbf{w}=-w\hat{\mathbf{x}}$ and $\frac{d\mathbf{w}}{dt'}=+\beta c\hat{\mathbf{x}}$ where $0<\beta<1$. That is, the charge is on the negative x axis and is moving in the $+x$ direction with a speed βc. Then we have

$$\begin{aligned}
\frac{d}{dt'}\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|^2 &= 2\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|\frac{d}{dt'}\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right| && (19)\\
&= \frac{d}{dt'}\left[\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\right] && (20)\\
&= -2\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\frac{d\mathbf{w}}{dt'} && (21)\\
&= -2\beta c\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\hat{\mathbf{x}} && (22)\\
\frac{d}{dt'}\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right| &= -\frac{\beta c\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\hat{\mathbf{x}}}{\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|} && (23)\\
&= -\frac{\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\mathbf{v}}{\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|} && (24)
\end{aligned}$$

where in the last line $\mathbf{v}\equiv\beta c\hat{\mathbf{x}}$ is the velocity of the charge. Therefore

$$\frac{df}{dt'}=1-\frac{\left(\mathbf{r}-\mathbf{w}\left(t'\right)\right)\cdot\mathbf{v}}{c\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|} \tag{25}$$

Returning to 12 we have $f\left(t'\right)=0$ when $t'=t_r$ so we can do the integral over the delta function to get

$$\begin{aligned}
V\left(\mathbf{r},t\right) &= \frac{q}{4\pi\epsilon_0}\int dt'\frac{1}{\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|}\delta\left(t'-\left(t-\frac{\left|\mathbf{r}-\mathbf{w}\left(t'\right)\right|}{c}\right)\right) && (26)\\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{\left|\mathbf{r}-\mathbf{w}\left(t_r\right)\right|\left(1-\frac{\left(\mathbf{r}-\mathbf{w}\left(t_r\right)\right)\cdot\mathbf{v}}{c\left|\mathbf{r}-\mathbf{w}\left(t_r\right)\right|}\right)} && (27)\\
&= \frac{qc}{4\pi\epsilon_0}\frac{1}{\left(c\left|\mathbf{r}-\mathbf{w}\left(t_r\right)\right|-\left(\mathbf{r}-\mathbf{w}\left(t_r\right)\right)\cdot\mathbf{v}\right)} && (28)
\end{aligned}$$

The current density for a moving point charge is just

$$\mathbf{J}=\rho\mathbf{v} \tag{29}$$

so the derivation of $\mathbf{A}$ from 2 follows exactly the same path and we get

$$\begin{aligned}
\mathbf{A}\left(\mathbf{r},t\right) &= \frac{\mu_0 qc}{4\pi}\frac{\mathbf{v}}{\left(c\left|\mathbf{r}-\mathbf{w}\left(t_r\right)\right|-\left(\mathbf{r}-\mathbf{w}\left(t_r\right)\right)\cdot\mathbf{v}\right)} && (30)\\
&= \frac{\mathbf{v}}{c^2}V\left(\mathbf{r},t\right) && (31)
\end{aligned}$$

These are the Liénard-Wiechert potentials for a moving point charge.

Griffiths gives a heuristic argument as to why the extra term $-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v}$ turns up in the denominator. The effect arises because of the perception of size of a moving object. If we see a metre stick coming directly at us with a speed v, we will perceive it to be slightly longer than it actually is, since the light from the far end of the stick left the stick when it was further away from us than the light from the near end. Although this argument does give the right answer and the argument doesn't depend ultimately on the size of the object approaching, I find the argument unsatisfying when applied to a point object, since I'd still expect that the effect should disappear in that case. The argument above, using delta functions, is a lot more abstract than the moving metre stick argument, but at least it shows rigorously how the effect arises.

Example. We have a point charge q moving in a circle of radius a in the xy plane at constant angular speed ω so that its position is given by

$$\mathbf{w}(t)=a\hat{\mathbf{x}}\cos\omega t+a\hat{\mathbf{y}}\sin\omega t \tag{32}$$

The velocity is

$$\mathbf{v}(t)=\frac{d\mathbf{w}}{dt}=-a\omega\hat{\mathbf{x}}\sin\omega t+a\omega\hat{\mathbf{y}}\cos\omega t \tag{33}$$

For an observation point $\mathbf{r}=z\hat{\mathbf{z}}$ the retarded time is

$$t_r=t-\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}=t-\frac{\sqrt{z^2+a^2}}{c} \tag{34}$$

This is independent of the charge's position, since it's always at the same distance from a point on the z axis. Also by direct calculation

$$(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v}=0 \tag{35}$$

so the potentials are

$$V(\mathbf{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c|\mathbf{r}-\mathbf{w}(t_r)|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \tag{36}$$

$$= \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{w}(t_r)|} \tag{37}$$

$$= \frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}} \tag{38}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{qa\omega}{4\pi\epsilon_0c^2\sqrt{z^2+a^2}}(-\hat{\mathbf{x}}\sin\omega t_r+\hat{\mathbf{y}}\cos\omega t_r) \tag{39}$$

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