

## LIÉNARD-WIECHERT POTENTIALS FOR A CHARGE MOVING WITH CONSTANT VELOCITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.14.

Griffiths shows in his example 10.3 that the Liénard-Wiechert potentials for a point charge  $q$  moving at constant velocity  $\mathbf{v}$  that passes through the origin at time  $t = 0$  are

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}} \quad (1)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{\sqrt{(c^2t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}} \quad (2)$$

These potentials can be expressed in a simpler form by defining the vector

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t \quad (3)$$

We can eliminate  $\mathbf{r}$  from 1 as follows.

$$\mathbf{R} \cdot \mathbf{v} = \mathbf{r} \cdot \mathbf{v} - v^2t \quad (4)$$

$$\mathbf{r} \cdot \mathbf{v} = \mathbf{R} \cdot \mathbf{v} + v^2t \quad (5)$$

$$R^2 = r^2 + v^2t^2 - 2\mathbf{r} \cdot \mathbf{v}t \quad (6)$$

$$r^2 = R^2 - v^2t^2 + 2\mathbf{r} \cdot \mathbf{v}t \quad (7)$$

$$= R^2 + v^2t^2 + 2\mathbf{R} \cdot \mathbf{v}t \quad (8)$$

$$(c^2t - \mathbf{r} \cdot \mathbf{v})^2 = (c^2t - \mathbf{R} \cdot \mathbf{v} - v^2t)^2 \quad (9)$$

$$(c^2 - v^2)(r^2 - c^2t^2) = (c^2 - v^2)(R^2 + v^2t^2 + 2\mathbf{R} \cdot \mathbf{v}t - c^2t^2) \quad (10)$$

Adding the last two RHSs together and cancelling terms, we get

$$(c^2t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2) = (\mathbf{R} \cdot \mathbf{v})^2 + (c^2 - v^2)R^2 \quad (11)$$

If  $\theta$  is the angle between  $\mathbf{R}$  and  $\mathbf{v}$ , then this becomes

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$$(\mathbf{R} \cdot \mathbf{v})^2 + (c^2 - v^2) R^2 = R^2 (c^2 - v^2 (1 - \cos^2 \theta)) \quad (12)$$

$$= R^2 c^2 \left( 1 - \frac{v^2}{c^2} \sin^2 \theta \right) \quad (13)$$

Inserting this back into 1 we get

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R \sqrt{\left( 1 - \frac{v^2}{c^2} \sin^2 \theta \right)}} \quad (14)$$

Note that  $R$  and  $\theta$  are both functions of time since they vary as the charge moves. For non-relativistic speeds,  $v \ll c$  and the formula reduces to the Coulomb potential from electrostatics:

$$V = \frac{q}{4\pi\epsilon_0 R} \quad (15)$$