

LIÉNARD-WIECHERT POTENTIAL FOR A CHARGE MOVING ON A HYPERBOLIC TRAJECTORY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.16.

We can now work out the Liénard-Wiechert potentials for a point charge moving on a hyperbolic trajectory. We're trying to find

$$(1) \quad V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r)) \cdot \mathbf{v})}$$

The motion is in one dimension, given by

$$(2) \quad \mathbf{w}(t) = \sqrt{b^2 + c^2 t^2} \hat{\mathbf{x}}$$

It's worth noting here that $w \geq b > 0$ for all times, and the velocity is

$$(3) \quad \mathbf{v}(t) = \frac{d\mathbf{w}}{dt} = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}} \hat{\mathbf{x}}$$

which is negative for $t < 0$ (when the particle is moving in from the right) and positive for $t > 0$ (when the particle is moving back out again).

We'll consider only observation points \mathbf{r} that lie on the x axis to the right of the particle's location, so the separation is

$$(4) \quad d(t) = r - w(t) = x - \sqrt{b^2 + c^2 t^2} > 0$$

Using this notation, the potential is

$$(5) \quad V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(cd(t_r) - \mathbf{d}(t_r) \cdot \mathbf{v})}$$

The retarded time is given by

$$(6) \quad d(t_r) = c(t - t_r)$$

$$(7) \quad x - \sqrt{b^2 + c^2 t_r^2} = c(t - t_r)$$

We can solve this by isolating the square root on one side and then squaring both sides, to get

$$(8) \quad t_r = \frac{x^2 - 2xct + c^2t^2 - b^2}{2c(ct - x)}$$

Note that for $t = 0$, as $x \rightarrow \infty$, $t_r \rightarrow -\infty$ which makes sense, since the further out on the x axis we place the observer, the further back in time we need to go to get a signal from the particle. We can substitute this back into 2 to get

$$(9) \quad w(t_r) = \frac{1}{2} \sqrt{\frac{(x^2 - 2xct + c^2t^2 + b^2)^2}{(ct - x)^2}}$$

Although the operand of the square root is a perfect square, we need to be careful when taking the square root to ensure we get the correct sign. Since $w > 0$ we can look at $t = 0$ as before, at which point

$$(10) \quad w(t_r) = \frac{1}{2} \sqrt{\frac{(x^2 + b^2)^2}{(-x)^2}} > 0$$

so we need to take the negative root of the operand to get $w > 0$. That is

$$(11) \quad w(t_r) = \frac{x^2 - 2xct + c^2t^2 + b^2}{2(x - tc)}$$

We can now calculate

$$(12) \quad d(t_r) = x - w(t_r)$$

$$(13) \quad = \frac{x^2 - c^2t^2 - b^2}{2(x - ct)}$$

after simplifying.

Now we need to find $\mathbf{v}(t_r)$. Substituting 8 into 3

$$(14) \quad \mathbf{v}(t_r) = -\frac{c(b^2 - x^2 + 2xct - c^2t^2)}{2(ct - x) \sqrt{b^2 + \frac{(b^2 - x^2 + 2xct - c^2t^2)^2}{4(-x + ct)^2}}}$$

Simplifying the operand of the square root, we get

$$(15) \quad \mathbf{v}(t_r) = -\frac{c(b^2 - x^2 + 2xct - c^2t^2)}{2(ct - x) \sqrt{\frac{(x^2 - 2xct + c^2t^2 + b^2)^2}{4(ct - x)^2}}}$$

Again, we need to take the correct sign when taking the square root. For $t = 0$ we get

$$(16) \quad \mathbf{v}(t_r) = c \frac{b^2 - x^2}{2x \sqrt{\frac{(x^2 + b^2)^2}{4(-x)^2}}}$$

For large x , the signal comes from the particle in the distant past when it was moving to the left, so we should have v negative in this case. This again requires taking the negative root, so we get

$$(17) \quad \mathbf{v}(t_r) = \frac{c(b^2 - x^2 + 2xct - c^2t^2)}{x^2 - 2xct + c^2t^2 + b^2}$$

Putting it all together, we get

$$(18) \quad cd(t_r) - \mathbf{d}(t_r) \cdot \mathbf{v} = \frac{1}{2} \frac{(-x^2 + c^2t^2 + b^2)c}{-x + ct} - \frac{1}{2} \frac{(-x^2 + c^2t^2 + b^2)c(b^2 - (-x + ct)^2)}{(-x + ct)(b^2 + (-x + ct)^2)}$$

$$(19) \quad = \frac{(-x + ct)(-x^2 + c^2t^2 + b^2)c}{x^2 - 2xct + c^2t^2 + b^2}$$

We've used Maple to do some of the algebra. Therefore the potential is

$$(20) \quad V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q(x^2 - 2xct + c^2t^2 + b^2)}{(-x + ct)(-x^2 + c^2t^2 + b^2)}$$

$$(21) \quad = \frac{1}{4\pi\epsilon_0} \frac{q((x - ct)^2 + b^2)}{(x - ct)(x^2 - c^2t^2 - b^2)}$$