FIELDS OF A MOVING POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.17.

We can use the Liénard-Wiechert potentialsfor a moving point charge to calculate the fields produced by the charge. The potentials are

$$V(\mathbf{r},t) = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(c\,|\mathbf{r}-\mathbf{w}(t_r)|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \tag{0.1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0 c}\frac{\mathbf{v}}{(c\,|\mathbf{r}-\mathbf{w}(t_r)|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \tag{0.2}$$

where $\mathbf{w}(t_r)$ is the position of the particle at the retarded time t_r and $\mathbf{v} = d\mathbf{w}/dt_r$ is the velocity at the same time.

I'll use the shorthand variable

$$\mathfrak{r} \equiv \mathbf{r}-\mathbf{w}(t_r) \tag{0.3}$$

in what follows (since I don't know of any Latex symbol to match the script-r used by Griffiths). Thus the potentials are written as

$$V(\mathbf{r},t) = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})} \tag{0.4}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0 c}\frac{\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})} \tag{0.5}$$

Given the potentials, we can work out the fields using

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \tag{0.6}$$

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{0.7}$$

Because of the convoluted dependence of the various quantities on the observer's location $\mathbf{r}$ and time t, these derivatives get quite involved. Griffiths works out ∇V in his section 10.3.2, with the result

$$\nabla V=\frac{qc}{4\pi\varepsilon_0\left(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}\right)^3}\left[\left(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}\right)\mathbf{v}-\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\boldsymbol{\mathfrak{r}}\right] \tag{0.8}$$

so we'll deal with $\frac{\partial\mathbf{A}}{\partial t}$ here.

First, it's useful to work out $\dot{t}_r\equiv dt_r/dt$. The retarded time t_r is defined implicitly by the equation

$$\mathfrak{r}=\left|\mathbf{r}-\mathbf{w}\left(t_r\right)\right|=c\left(t-t_r\right) \tag{0.9}$$

We start with

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{\mathfrak{r}}} &= \frac{1}{2\mathfrak{r}}\frac{\partial}{\partial t}\left(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{\mathfrak{r}}\right) &(0.10)\\
&= \frac{1}{2\mathfrak{r}}\frac{\partial}{\partial t}\left(r^2-2\mathbf{r}\cdot\mathbf{w}+w^2\right) &(0.11)\\
&= \frac{1}{2\mathfrak{r}}\left(-2\mathbf{r}\cdot\frac{d\mathbf{w}}{dt_r}\dot{t}_r+2\mathbf{w}\cdot\frac{d\mathbf{w}}{dt_r}\dot{t}_r\right) &(0.12)\\
&= \frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot\left(\mathbf{w}-\mathbf{r}\right) &(0.13)\\
&= -\frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\mathfrak{r}} &(0.14)
\end{aligned}$$

From the RHS of 0.9 we get

$$c\frac{d}{dt}\left(t-t_r\right)=c\left(1-\dot{t}_r\right) \tag{0.15}$$

Combining these two equations we get

$$\begin{aligned}
c\left(1-\dot{t}_r\right) &= -\frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\mathfrak{r}} &(0.16)\\
\dot{t}_r &= \frac{\mathfrak{r}c}{\mathfrak{r}c-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}} &(0.17)\\
&= \frac{\mathfrak{r}c}{\boldsymbol{\mathfrak{r}}\cdot\mathbf{u}} &(0.18)
\end{aligned}$$

where

$$\mathbf{u}\equiv c\hat{\boldsymbol{\mathfrak{r}}}-\mathbf{v} \tag{0.19}$$

Returning to 0.5 we now have

(0.20)
$$\frac{4\pi c\varepsilon_0}{q}\frac{\partial\mathbf{A}}{\partial t}=\frac{\partial\mathbf{v}}{\partial t}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\mathbf{v}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\frac{\partial}{\partial t}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})$$

(0.21)
$$=\frac{\partial\mathbf{v}}{\partial t_r}\dot{t}_r\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\mathbf{v}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}c\frac{\partial\mathfrak{r}}{\partial t}-\frac{\partial\boldsymbol{\mathfrak{r}}}{\partial t}\cdot\mathbf{v}-\boldsymbol{\mathfrak{r}}\cdot\frac{\partial\mathbf{v}}{\partial t_r}\dot{t}_r$$

The derivative

(0.22)
$$\frac{\partial\mathbf{v}}{\partial t_r}\equiv\mathbf{a}(t_r)$$

is the acceleration of the particle at the retarded time. The other two derivatives are, from 0.9

(0.23)
$$\frac{\partial\mathfrak{r}}{\partial t}=c(1-\dot{t}_r)$$

(0.24)
$$\frac{\partial\boldsymbol{\mathfrak{r}}}{\partial t}=-\frac{d\mathbf{w}}{dt_r}\dot{t}_r=-\mathbf{v}\dot{t}_r$$

Therefore

(0.25)
$$\frac{4\pi\varepsilon_0 c}{q}\frac{\partial\mathbf{A}}{\partial t}=\dot{t}_r\left[\frac{\mathbf{a}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\frac{\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\left(-c^2+v^2-\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right]-\frac{c^2\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}$$

(0.26)
$$=\frac{\mathfrak{r}c}{\mathfrak{r}c-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}}\left[\frac{\mathbf{a}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\frac{\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\left(-c^2+v^2-\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right]-\frac{c^2\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}$$

(0.27)
$$=\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\mathfrak{r}c\left(\mathbf{a}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})+\mathbf{v}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right)-c^2\mathbf{v}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})\right]$$

(0.28)
$$=\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\left(\mathfrak{r}c\mathbf{a}-c^2\mathbf{v}\right)(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})+\mathfrak{r}c\mathbf{v}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right]$$

(0.29)
$$\frac{\partial\mathbf{A}}{\partial t}=\frac{qc}{4\pi\varepsilon_0(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\left(\frac{\mathfrak{r}\mathbf{a}}{c}-\mathbf{v}\right)(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})+\frac{\mathfrak{r}\mathbf{v}}{c}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right]$$

Combining this with 0.8 and inserting into 0.6 gives the result quoted in Griffiths as equation 10.65:

$$\mathbf{E}(\mathbf{r},t)=\frac{q\mathfrak{r}}{4\pi\epsilon_0(\mathfrak{r}\cdot\mathbf{u})^3}\left[\left(c^2-v^2\right)\mathbf{u}+\mathfrak{r}\times(\mathbf{u}\times\mathbf{a})\right] \tag{0.30}$$

For reference, the magnetic field is

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A} \tag{0.31}$$

$$= \frac{1}{c}\hat{\mathfrak{r}}\times\mathbf{E}(\mathbf{r},t) \tag{0.32}$$

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