FIELDS OF A MOVING POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.17.

We can use the Liénard-Wiechert potentialsfor a moving point charge to calculate the fields produced by the charge. The potentials are

$$V(\mathbf{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c|\mathbf{r}-\mathbf{w}(t_r)|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \quad (1)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c}\frac{\mathbf{v}}{(c|\mathbf{r}-\mathbf{w}(t_r)|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \quad (2)$$

where $\mathbf{w}(t_r)$ is the position of the particle at the retarded time t_r and $\mathbf{v} = d\mathbf{w}/dt_r$ is the velocity at the same time.

I'll use the shorthand variable

$$\mathfrak{r} \equiv \mathbf{r}-\mathbf{w}(t_r) \quad (3)$$

in what follows (since I don't know of any Latex symbol to match the script-r used by Griffiths). Thus the potentials are written as

$$V(\mathbf{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c\mathfrak{r}-\mathfrak{r}\cdot\mathbf{v})} \quad (4)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c}\frac{\mathbf{v}}{(c\mathfrak{r}-\mathfrak{r}\cdot\mathbf{v})} \quad (5)$$

Given the potentials, we can work out the fields using

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \quad (6)$$

$$\mathbf{B} = \nabla\times\mathbf{A} \quad (7)$$

Because of the convoluted dependence of the various quantities on the observer's location $\mathbf{r}$ and time t, these derivatives get quite involved. Griffiths works out ∇V in his section 10.3.2, with the result

$$\nabla V = \frac{qc}{4\pi\epsilon_0(c\mathfrak{r}-\mathfrak{r}\cdot\mathbf{v})^3}\left[(c\mathfrak{r}-\mathfrak{r}\cdot\mathbf{v})\mathbf{v}-\left(c^2-v^2+\mathfrak{r}\cdot\mathbf{a}\right)\mathfrak{r}\right] \quad (8)$$

so we'll deal with $\frac{\partial\mathbf{A}}{\partial t}$ here.

First, it's useful to work out $\dot{t}_r \equiv dt_r/dt$. The retarded time t_r is defined implicitly by the equation

$$\mathfrak{r} = |\mathbf{r}-\mathbf{w}(t_r)| = c(t-t_r) \tag{9}$$

We start with

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{\mathfrak{r}}} &= \frac{1}{2\mathfrak{r}}\frac{\partial}{\partial t}(\boldsymbol{\mathfrak{r}}\cdot\boldsymbol{\mathfrak{r}}) && (10)\\
&= \frac{1}{2\mathfrak{r}}\frac{\partial}{\partial t}\left(r^2-2\mathbf{r}\cdot\mathbf{w}+w^2\right) && (11)\\
&= \frac{1}{2\mathfrak{r}}\left(-2\mathbf{r}\cdot\frac{d\mathbf{w}}{dt_r}\dot{t}_r+2\mathbf{w}\cdot\frac{d\mathbf{w}}{dt_r}\dot{t}_r\right) && (12)\\
&= \frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot(\mathbf{w}-\mathbf{r}) && (13)\\
&= -\frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\mathfrak{r}} && (14)
\end{aligned}$$

From the RHS of 9 we get

$$c\frac{d}{dt}(t-t_r) = c(1-\dot{t}_r) \tag{15}$$

Combining these two equations we get

$$\begin{aligned}
c(1-\dot{t}_r) &= -\frac{1}{\mathfrak{r}}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\mathfrak{r}} && (16)\\
\dot{t}_r &= \frac{\mathfrak{r}c}{\mathfrak{r}c-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}} && (17)\\
&= \frac{\mathfrak{r}c}{\boldsymbol{\mathfrak{r}}\cdot\mathbf{u}} && (18)
\end{aligned}$$

where

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathfrak{r}}}-\mathbf{v} \tag{19}$$

Returning to 5 we now have

$$\frac{4\pi c\epsilon_0}{q}\frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{v}}{\partial t}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\mathbf{v}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\frac{\partial}{\partial t}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}) \tag{20}$$

$$= \frac{\partial\mathbf{v}}{\partial t_r}\dot{t}_r\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})}-\mathbf{v}\frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}c\frac{\partial\mathfrak{r}}{\partial t}-\frac{\partial\boldsymbol{\mathfrak{r}}}{\partial t}\cdot\mathbf{v}-\boldsymbol{\mathfrak{r}}\cdot\frac{\partial\mathbf{v}}{\partial t_r}\dot{t}_r \tag{21}$$

The derivative

$$\frac{\partial \mathbf{v}}{\partial t_r} \equiv \mathbf{a}(t_r) \tag{22}$$

is the acceleration of the particle at the retarded time. The other two derivatives are, from 9

$$\frac{\partial \mathfrak{r}}{\partial t} = c(1-\dot{t}_r) \tag{23}$$

$$\frac{\partial \boldsymbol{\mathfrak{r}}}{\partial t} = -\frac{d\mathbf{w}}{dt_r}\dot{t}_r = -\mathbf{v}\dot{t}_r \tag{24}$$

Therefore

$$\frac{4\pi\epsilon_0 c}{q}\frac{\partial \mathbf{A}}{\partial t} = \dot{t}_r\left[\frac{\mathbf{a}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})} - \frac{\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\left(-c^2+v^2-\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right] - \frac{c^2\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2} \tag{25}$$

$$= \frac{\mathfrak{r}c}{\mathfrak{r}c-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}}\left[\frac{\mathbf{a}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})} - \frac{\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2}\left(-c^2+v^2-\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right] - \frac{c^2\mathbf{v}}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^2} \tag{26}$$

$$= \frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\mathfrak{r}c\left(\mathbf{a}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})+\mathbf{v}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right) - c^2\mathbf{v}(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})\right] \tag{27}$$

$$= \frac{1}{(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\left(\mathfrak{r}c\mathbf{a}-c^2\mathbf{v}\right)(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}) + \mathfrak{r}c\mathbf{v}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right] \tag{28}$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v})^3}\left[\left(\frac{\mathfrak{r}\mathbf{a}}{c}-\mathbf{v}\right)(c\mathfrak{r}-\boldsymbol{\mathfrak{r}}\cdot\mathbf{v}) + \frac{\mathfrak{r}\mathbf{v}}{c}\left(c^2-v^2+\boldsymbol{\mathfrak{r}}\cdot\mathbf{a}\right)\right] \tag{29}$$

Combining this with 8 and inserting into 6 gives the result quoted in Griffiths as equation 10.65:

$$\mathbf{E}(\mathbf{r},t) = \frac{q\mathfrak{r}}{4\pi\epsilon_0(\boldsymbol{\mathfrak{r}}\cdot\mathbf{u})^3}\left[\left(c^2-v^2\right)\mathbf{u} + \boldsymbol{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a})\right] \tag{30}$$

For reference, the magnetic field is

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A} \tag{31}$$

$$= \frac{1}{c}\hat{\boldsymbol{\mathfrak{r}}}\times\mathbf{E}(\mathbf{r},t) \tag{32}$$

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