

## FIELDS OF A MOVING POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problems 10.17.

We can use the Liénard-Wiechert potentials for a moving point charge to calculate the fields produced by the charge. The potentials are

$$(1) \quad V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r)) \cdot \mathbf{v})}$$
$$(2) \quad \mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{v}}{(c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r)) \cdot \mathbf{v})}$$

where  $\mathbf{w}(t_r)$  is the position of the particle at the retarded time  $t_r$  and  $\mathbf{v} = d\mathbf{w}/dt_r$  is the velocity at the same time.

I'll use the shorthand variable

$$(3) \quad \mathbf{r} \equiv \mathbf{r} - \mathbf{w}(t_r)$$

in what follows (since I don't know of any Latex symbol to match the script- $r$  used by Griffiths). Thus the potentials are written as

$$(4) \quad V(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{(ct - \mathbf{r} \cdot \mathbf{v})}$$
$$(5) \quad \mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{v}}{(ct - \mathbf{r} \cdot \mathbf{v})}$$

Given the potentials, we can work out the fields using

$$(6) \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$
$$(7) \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Because of the convoluted dependence of the various quantities on the observer's location  $\mathbf{r}$  and time  $t$ , these derivatives get quite involved. Griffiths works out  $\nabla V$  in his section 10.3.2, with the result

$$(8) \quad \nabla V = \frac{qc}{4\pi\epsilon_0 (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) \mathbf{v} - (c^2 - v^2 + \mathbf{r} \cdot \mathbf{a}) \mathbf{r}]$$

so we'll deal with  $\frac{\partial \mathbf{A}}{\partial t}$  here.

First, it's useful to work out  $\dot{t}_r \equiv dt_r/dt$ . The retarded time  $t_r$  is defined implicitly by the equation

$$(9) \quad \tau = |\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r)$$

We start with

$$(10) \quad \frac{\partial}{\partial t} \sqrt{\mathbf{r} \cdot \mathbf{r}} = \frac{1}{2\mathbf{r}} \frac{\partial}{\partial t} (\mathbf{r} \cdot \mathbf{r})$$

$$(11) \quad = \frac{1}{2\mathbf{r}} \frac{\partial}{\partial t} (r^2 - 2\mathbf{r} \cdot \mathbf{w} + w^2)$$

$$(12) \quad = \frac{1}{2\mathbf{r}} \left( -2\mathbf{r} \cdot \frac{d\mathbf{w}}{dt_r} \dot{t}_r + 2\mathbf{w} \cdot \frac{d\mathbf{w}}{dt_r} \dot{t}_r \right)$$

$$(13) \quad = \frac{1}{\mathbf{r}} \dot{t}_r \mathbf{v} \cdot (\mathbf{w} - \mathbf{r})$$

$$(14) \quad = -\frac{1}{\mathbf{r}} \dot{t}_r \mathbf{v} \cdot \mathbf{r}$$

From the RHS of 9 we get

$$(15) \quad c \frac{d}{dt} (t - t_r) = c(1 - \dot{t}_r)$$

Combining these two equations we get

$$(16) \quad c(1 - \dot{t}_r) = -\frac{1}{\mathbf{r}} \dot{t}_r \mathbf{v} \cdot \mathbf{r}$$

$$(17) \quad \dot{t}_r = \frac{\mathbf{r}c}{\mathbf{r}c - \mathbf{r} \cdot \mathbf{v}}$$

$$(18) \quad = \frac{\mathbf{r}c}{\mathbf{r} \cdot \mathbf{u}}$$

where

$$(19) \quad \mathbf{u} \equiv c\hat{\mathbf{r}} - \mathbf{v}$$

Returning to 5 we now have

$$(20) \quad \frac{4\pi c\epsilon_0}{q} \frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \mathbf{v} \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} \frac{\partial}{\partial t} (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})$$

$$(21) \quad = \frac{\partial \mathbf{v}}{\partial t_r} i_r \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \mathbf{v} \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} c \frac{\partial \mathbf{r}}{\partial t} - \frac{\partial \mathbf{r}}{\partial t} \cdot \mathbf{v} - \mathbf{r} \cdot \frac{\partial \mathbf{v}}{\partial t_r} i_r$$

The derivative

$$(22) \quad \frac{\partial \mathbf{v}}{\partial t_r} \equiv \mathbf{a}(t_r)$$

is the acceleration of the particle at the retarded time. The other two derivatives are, from 9

$$(23) \quad \frac{\partial \mathbf{r}}{\partial t} = c(1 - i_r)$$

$$(24) \quad \frac{\partial \mathbf{r}}{\partial t} = -\frac{d\mathbf{w}}{dt_r} i_r = -\mathbf{v} i_r$$

Therefore

$$(25) \quad \frac{4\pi\epsilon_0 c}{q} \frac{\partial \mathbf{A}}{\partial t} = i_r \left[ \frac{\mathbf{a}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \frac{\mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} (-c^2 + v^2 - \mathbf{r} \cdot \mathbf{a}) \right] - \frac{c^2 \mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2}$$

$$(26) \quad = \frac{\mathbf{r}c}{\mathbf{r}c - \mathbf{r} \cdot \mathbf{v}} \left[ \frac{\mathbf{a}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})} - \frac{\mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2} (-c^2 + v^2 - \mathbf{r} \cdot \mathbf{a}) \right] - \frac{c^2 \mathbf{v}}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^2}$$

$$(27) \quad = \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [\mathbf{r}c (\mathbf{a}(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \mathbf{v}(c^2 - v^2 + \mathbf{r} \cdot \mathbf{a})) - c^2 \mathbf{v}(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})]$$

$$(28) \quad = \frac{1}{(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} [(\mathbf{r}c\mathbf{a} - c^2 \mathbf{v})(c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \mathbf{r}c\mathbf{v}(c^2 - v^2 + \mathbf{r} \cdot \mathbf{a})]$$

$$(29) \quad \frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0 (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v})^3} \left[ \left( \frac{\mathbf{r}\mathbf{a}}{c} - \mathbf{v} \right) (c\mathbf{r} - \mathbf{r} \cdot \mathbf{v}) + \frac{\mathbf{r}\mathbf{v}}{c} (c^2 - v^2 + \mathbf{r} \cdot \mathbf{a}) \right]$$

Combining this with 8 and inserting into 6 gives the result quoted in Griffiths as equation 10.65:

$$(30) \quad \mathbf{E}(\mathbf{r}, t) = \frac{q\mathbf{r}}{4\pi\epsilon_0(\mathbf{r}\cdot\mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})]$$

For reference, the magnetic field is

$$(31) \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}$$

$$(32) \quad = \frac{1}{c} \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, t)$$

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