

## FIELDS DUE TO A MOVING LINEAR CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.19.

In his example 10.4, Griffiths works out the fields due to a point charge moving with constant velocity  $\mathbf{v}$ . They are

$$(0.1) \quad \mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$$

where

$$(0.2) \quad \mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

is the vector from the particle's present (not retarded) position to the observer (assuming the particle passes through the origin at  $t = 0$ ) and  $\theta$  is the angle between  $\mathbf{R}$  and  $\mathbf{v}$ . We can use this formula to rederive the equation for the electric field due to an infinite line charge with linear charge density  $\lambda$ . From electrostatics, we know the field is given by

$$(0.3) \quad \mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 z}$$

where  $z$  is the perpendicular distance from the line (wire). Let's see if we can get the same result using the formula above.

The field due to a small segment of the wire of length  $dx$  at position is that due to a point charge  $\lambda dx$ . For an observation point at  $\mathbf{r}$ , the length of  $\mathbf{R}$  is

$$(0.4) \quad R = \sqrt{z^2 + x^2}$$

and since the velocity is parallel to the wire, we have

$$(0.5) \quad \sin \theta = \frac{z}{\sqrt{z^2 + x^2}}$$

Since  $\mathbf{E}$  is parallel to  $\mathbf{R}$ , by symmetry the components of  $\mathbf{E}$  parallel to the wire will cancel out, since there will be equal and opposite contributions from points  $\pm x$ . The perpendicular component is  $\mathbf{E} \sin \theta$  so the total field is

$$(0.6) \quad \mathbf{E} = \frac{\lambda}{4\pi\epsilon_0} \left(1 - \frac{v^2}{c^2}\right) \int_{-\infty}^{\infty} \frac{\sin \theta dx}{(1 - v^2 \sin^2 \theta / c^2)^{3/2} (z^2 + x^2)} \hat{\mathbf{s}}$$

where the  $s$  direction is radial. We can convert this to an integral over  $\theta$  by noting that

$$(0.7) \quad \cos \theta d\theta = -\frac{xz}{(z^2 + x^2)^{3/2}} dx$$

$$(0.8) \quad \frac{dx}{z^2 + x^2} = -\frac{\sqrt{z^2 + x^2}}{xz} \left(-\frac{xz}{(z^2 + x^2)^{3/2}} dx\right)$$

$$(0.9) \quad = -\frac{\sqrt{z^2 + x^2}}{xz} \cos \theta d\theta$$

But

$$(0.10) \quad \cos \theta = -\frac{x}{\sqrt{z^2 + x^2}}$$

so

$$(0.11) \quad \frac{dx}{z^2 + x^2} = \frac{d\theta}{z}$$

$$(0.12) \quad \frac{\lambda}{4\pi\epsilon_0} \left(1 - \frac{v^2}{c^2}\right) \int_{-\infty}^{\infty} \frac{\sin \theta dx}{(1 - v^2 \sin^2 \theta / c^2)^{3/2} (z^2 + x^2)} \hat{\mathbf{s}} = \frac{\lambda}{4\pi\epsilon_0 z} \left(1 - \frac{v^2}{c^2}\right) \int_0^\pi \frac{\sin \theta d\theta}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \hat{\mathbf{s}}$$

The integral can be evaluated using Maple, and we get

$$(0.13) \quad \int_0^\pi \frac{\sin \theta d\theta}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \hat{\mathbf{s}} = \left. \frac{-\cos \theta}{\left(1 - \frac{v^2}{c^2}\right) \sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} \hat{\mathbf{s}} \right|_0^\pi$$

$$(0.14) \quad = \frac{2}{1 - \frac{v^2}{c^2}} \hat{\mathbf{s}}$$

so we get back the correct field

$$(0.15) \quad \mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 z} \hat{\mathbf{s}}$$

The magnetic field of a point charge is given by Griffiths as

$$(0.16) \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}(\mathbf{r}, t)$$

Since  $\mathbf{v}$  is a constant, the total magnetic field can be found from the same integral as above. Its direction is given by  $\hat{\mathbf{x}} \times \hat{\mathbf{s}} = \hat{\phi}$  which circles the wire in a direction given by the usual right-hand rule. Since  $\lambda \mathbf{v} = \mathbf{I}$  (the current), we get

$$(0.17) \quad \mathbf{B} = \frac{I}{2\pi\epsilon_0 c^2 z} \hat{\phi}$$

$$(0.18) \quad = \frac{\mu_0 I}{2\pi z} \hat{\phi}$$

which agrees with the magnetostatic formula using Ampère's law.

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