FIELDS DUE TO A MOVING LINEAR CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.19.

In his example 10.4, Griffiths works out the fields due to a point charge moving with constant velocity $\mathbf{v}$. They are

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} \tag{1}$$

where

$$\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t \tag{2}$$

is the vector from the particle's present (not retarded) position to the observer (assuming the particle passes through the origin at $t=0$) and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$. We can use this formula to rederive the equation for the electric field due to an infinite line charge with linear charge density λ. From electrostatics, we know the field is given by

$$\mathbf{E}=\frac{\lambda}{2\pi\varepsilon_0 z} \tag{3}$$

where z is the perpendicular distance from the line (wire). Let's see if we can get the same result using the formula above.

The field due to a small segment of the wire of length dx at position is that due to a point charge λdx. For an observation point at $\mathbf{r}$, the length of $\mathbf{R}$ is

$$R=\sqrt{z^2+x^2} \tag{4}$$

and since the velocity is parallel to the wire, we have

$$\sin\theta=\frac{z}{\sqrt{z^2+x^2}} \tag{5}$$

Since $\mathbf{E}$ is parallel to $\mathbf{R}$, by symmetry the components of $\mathbf{E}$ parallel to the wire will cancel out, since there will be equal and opposite contributions from points $\pm x$. The perpendicular component is $\mathbf{E}\sin\theta$ so the total field is

$$\mathbf{E}=\frac{\lambda}{4\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)\int_{-\infty}^{\infty}\frac{\sin\theta dx}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{s}}}{(z^2+x^2)} \tag{6}$$

where the s direction is radial. We can convert this to an integral over θ by noting that

$$\begin{aligned}\cos\theta d\theta &= -\frac{xz}{(z^2+x^2)^{3/2}}dx &(7)\\ \frac{dx}{z^2+x^2} &= -\frac{\sqrt{z^2+x^2}}{xz}\left(-\frac{xz}{(z^2+x^2)^{3/2}}dx\right) &(8)\\ &= -\frac{\sqrt{z^2+x^2}}{xz}\cos\theta d\theta &(9)\end{aligned}$$

But

$$\cos\theta=-\frac{x}{\sqrt{z^2+x^2}} \tag{10}$$

so

$$\frac{dx}{z^2+x^2}=\frac{d\theta}{z} \tag{11}$$

$$\frac{\lambda}{4\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)\int_{-\infty}^{\infty}\frac{\sin\theta dx}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{s}}}{(z^2+x^2)}=\frac{\lambda}{4\pi\varepsilon_0 z}\left(1-\frac{v^2}{c^2}\right)\int_0^{\pi}\frac{\sin\theta d\theta}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\hat{\mathbf{s}} \tag{12}$$

The integral can be evaluated using Maple, and we get

$$\begin{aligned}\int_0^{\pi}\frac{\sin\theta d\theta}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\hat{\mathbf{s}} &= \left.\frac{-\cos\theta}{\left(1-\frac{v^2}{c^2}\right)\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}\hat{\mathbf{s}}\right|_0^{\pi} &(13)\\ &= \frac{2}{1-\frac{v^2}{c^2}}\hat{\mathbf{s}} &(14)\end{aligned}$$

so we get back the correct field

$$\mathbf{E}=\frac{\lambda}{2\pi\varepsilon_{0}z}\hat{\mathbf{s}} \tag{15}$$

The magnetic field of a point charge is given by Griffiths as

$$\mathbf{B}\left(\mathbf{r},t\right)=\frac{1}{c^{2}}\mathbf{v}\times\mathbf{E}\left(\mathbf{r},t\right) \tag{16}$$

Since $\mathbf{v}$ is a constant, the total magnetic field can be found from the same integral as above. Its direction is given by $\hat{\mathbf{x}}\times\hat{\mathbf{s}}=\hat{\phi}$ which circles the wire in a direction given by the usual right-hand rule. Since $\lambda\mathbf{v}=\mathbf{I}$ (the current), we get

$$\begin{aligned}\mathbf{B} &= \frac{I}{2\pi\varepsilon_{0}c^{2}z}\hat{\phi} && (17)\\ &= \frac{\mu_{0}I}{2\pi z}\hat{\phi} && (18)\end{aligned}$$

which agrees with the magnetostatic formula using Ampère's law.

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