FIELDS OF A POINT CHARGE MOVING IN A CIRCLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.20.

We can now calculate the electric and magnetic fields due to a point charge moving in a circle of radius a at constant angular velocity ω. The charge's position and velocity are given by

$$\mathbf{w}(t) = a\hat{\mathbf{x}}\cos\omega t + a\hat{\mathbf{y}}\sin\omega t \tag{1}$$

$$\mathbf{v}(t) = -a\omega\hat{\mathbf{x}}\sin\omega t + a\omega\hat{\mathbf{y}}\cos\omega t \tag{2}$$

The calculation is easier if we use cylindrical coordinates, where

$$\hat{\mathbf{s}}(t) = \hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t \tag{3}$$

$$\hat{\phi}(t) = -\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t \tag{4}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{5}$$

Here, s is the radial coordinate and ϕ is the angular coordinate, and their unit vectors are both functions of time. Using these coordinates, we have

$$\mathbf{w} = a\hat{\mathbf{s}} \tag{6}$$

$$\mathbf{v} = a\omega\hat{\phi} \tag{7}$$

The fields of a moving point charge are

$$\mathbf{E}(\mathbf{r},t) = \frac{q\mathfrak{r}}{4\pi\epsilon_0(\mathfrak{r}\cdot\mathbf{u})^3}\left[\left(c^2-v^2\right)\mathbf{u}+\mathfrak{r}\times(\mathbf{u}\times\mathbf{a})\right] \tag{8}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathfrak{r}}\times\mathbf{E}(\mathbf{r},t) \tag{9}$$

where

$$\begin{aligned} \boldsymbol{\mathfrak{r}} &\equiv \mathbf{r}-\mathbf{w}(t_r) && (10)\\ \mathbf{u} &\equiv c\hat{\boldsymbol{\mathfrak{r}}}-\mathbf{v}(t_r) && (11)\\ \mathbf{a} &= \dot{\mathbf{v}}(t_r) && (12) \end{aligned}$$

For the fields at the centre of the circle $\mathbf{r}=0$ so we get

$$\begin{aligned} \boldsymbol{\mathfrak{r}} &= -\mathbf{w}(t_r) && (13)\\ &= -a\hat{\mathbf{s}}(t_r) && (14)\\ \mathbf{u} &= -c\hat{\mathbf{s}}-a\omega\hat{\phi} && (15)\\ \mathbf{a} &= -a\omega^2\hat{\mathbf{s}} && (16)\\ \mathbf{u}\times\mathbf{a} &= -a^2\omega^3\hat{\mathbf{z}} && (17)\\ \boldsymbol{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a}) &= -a^3\omega^3\hat{\phi} && (18)\\ \left(c^2-v^2\right)\mathbf{u}+\boldsymbol{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a}) &= \left(c^2-v^2\right)\left(-c\hat{\mathbf{s}}-a\omega\hat{\phi}\right)-a^3\omega^3\hat{\phi} && (19)\\ &= \left(ca^2\omega^2-c^3\right)\hat{\mathbf{s}}-c^2a\omega\hat{\phi} && (20)\\ \boldsymbol{\mathfrak{r}}\cdot\mathbf{u} &= ac && (21) \end{aligned}$$

Putting all this together, we get

$$\begin{aligned} \mathbf{E}(0,t) &= \frac{q}{4\pi\epsilon_0 a^2c^2}\left(\left(a^2\omega^2-c^2\right)\hat{\mathbf{s}}(t_r)-ca\omega\hat{\phi}(t_r)\right) && (22)\\ \mathbf{B}(0,t) &= \frac{q}{4\pi\epsilon_0 a^3c^3}(-a\hat{\mathbf{s}})\times\left(\left(a^2\omega^2-c^2\right)\hat{\mathbf{s}}-ca\omega\hat{\phi}\right) && (23)\\ &= \frac{q\omega}{4\pi\epsilon_0 ac^2}\hat{\mathbf{z}} && (24)\\ &= \frac{\mu_0 q\omega}{4\pi a}\hat{\mathbf{z}} && (25) \end{aligned}$$

The electric field depends on time, but the magnetic field is constant.

We can use this result to find the magnetic field at the centre due to a steady current caused by a line charge density of λ moving at angular velocity ω. In that case each infinitesimal segment of the loop has length $a\,d\phi$ so $q=\lambda a\,d\phi$. The current is $I=\lambda a\omega$ so the field is

$$\begin{aligned} \mathbf{B} &= \int_0^{2\pi}\frac{\mu_0\lambda a\omega\,d\phi}{4\pi a}\hat{\mathbf{z}} && (26)\\ &= \frac{\mu_0 I}{2a}\hat{\mathbf{z}} && (27) \end{aligned}$$

In his example 5.6 Griffiths gives the magnetic field along the z axis of a uniform circular current, and his result is

$$\mathbf{B} = \frac{\mu_0 a^2}{2\left(a^2 + z^2\right)^{3/2}}\hat{\mathbf{z}} \tag{28}$$

which reduces to our result at the centre of the circle where $z = 0$.