

## FIELDS OF A POINT CHARGE MOVING IN A CIRCLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.20.

We can now calculate the electric and magnetic fields due to a point charge moving in a circle of radius  $a$  at constant angular velocity  $\omega$ . The charge's position and velocity are given by

$$(0.1) \quad \mathbf{w}(t) = a\hat{\mathbf{x}}\cos\omega t + a\hat{\mathbf{y}}\sin\omega t$$

$$(0.2) \quad \mathbf{v}(t) = -a\omega\hat{\mathbf{x}}\sin\omega t + a\omega\hat{\mathbf{y}}\cos\omega t$$

The calculation is easier if we use cylindrical coordinates, where

$$(0.3) \quad \hat{\mathbf{s}}(t) = \hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t$$

$$(0.4) \quad \hat{\phi}(t) = -\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t$$

$$(0.5) \quad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Here,  $s$  is the radial coordinate and  $\phi$  is the angular coordinate, and their unit vectors are both functions of time. Using these coordinates, we have

$$(0.6) \quad \mathbf{w} = a\hat{\mathbf{s}}$$

$$(0.7) \quad \mathbf{v} = a\omega\hat{\phi}$$

The fields of a moving point charge are

$$(0.8) \quad \mathbf{E}(\mathbf{r}, t) = \frac{q\mathbf{r}}{4\pi\epsilon_0(\mathbf{r}\cdot\mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})]$$

$$(0.9) \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\mathbf{t}} \times \mathbf{E}(\mathbf{r}, t)$$

where

$$(0.10) \quad \mathbf{r} \equiv \mathbf{r} - \mathbf{w}(t_r)$$

$$(0.11) \quad \mathbf{u} \equiv c\hat{\mathbf{t}} - \mathbf{v}(t_r)$$

$$(0.12) \quad \mathbf{a} = \dot{\mathbf{v}}(t_r)$$

For the fields at the centre of the circle  $\mathbf{r} = 0$  so we get

$$(0.13) \quad \mathbf{r} = -\mathbf{w}(t_r)$$

$$(0.14) \quad = -a\hat{\mathbf{s}}(t_r)$$

$$(0.15) \quad \mathbf{u} = -c\hat{\mathbf{s}} - a\omega\hat{\phi}$$

$$(0.16) \quad \mathbf{a} = -a\omega^2\hat{\mathbf{s}}$$

$$(0.17) \quad \mathbf{u} \times \mathbf{a} = -a^2\omega^3\hat{\mathbf{z}}$$

$$(0.18) \quad \mathbf{r} \times (\mathbf{u} \times \mathbf{a}) = -a^3\omega^3\hat{\phi}$$

$$(0.19) \quad (c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a}) = (c^2 - v^2)(-c\hat{\mathbf{s}} - a\omega\hat{\phi}) - a^3\omega^3\hat{\phi}$$

$$(0.20) \quad = (ca^2\omega^2 - c^3)\hat{\mathbf{s}} - c^2a\omega\hat{\phi}$$

$$(0.21) \quad \mathbf{r} \cdot \mathbf{u} = ac$$

Putting all this together, we get

$$(0.22) \quad \mathbf{E}(0, t) = \frac{q}{4\pi\epsilon_0 a^2 c^2} ((a^2\omega^2 - c^2)\hat{\mathbf{s}}(t_r) - ca\omega\hat{\phi}(t_r))$$

$$(0.23) \quad \mathbf{B}(0, t) = \frac{q}{4\pi\epsilon_0 a^3 c^3} (-a\hat{\mathbf{s}}) \times ((a^2\omega^2 - c^2)\hat{\mathbf{s}} - ca\omega\hat{\phi})$$

$$(0.24) \quad = \frac{q\omega}{4\pi\epsilon_0 ac^2} \hat{\mathbf{z}}$$

$$(0.25) \quad = \frac{\mu_0 q \omega}{4\pi a} \hat{\mathbf{z}}$$

The electric field depends on time, but the magnetic field is constant.

We can use this result to find the magnetic field at the centre due to a steady current caused by a line charge density of  $\lambda$  moving at angular velocity  $\omega$ . In that case each infinitesimal segment of the loop has length  $a d\phi$  so  $q = \lambda a d\phi$ . The current is  $I = \lambda a \omega$  so the field is

$$(0.26) \quad \mathbf{B} = \int_0^{2\pi} \frac{\mu_0 \lambda a \omega d\phi}{4\pi a} \hat{\mathbf{z}}$$

$$(0.27) \quad = \frac{\mu_0 I}{2a} \hat{\mathbf{z}}$$

In his example 5.6 Griffiths gives the magnetic field along the  $z$  axis of a uniform circular current, and his result is

$$(0.28) \quad \mathbf{B} = \frac{\mu_0 a^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

which reduces to our result at the centre of the circle where  $z = 0$ .