

FIELDS OF A POINT CHARGE MOVING IN A CIRCLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.20.

We can now calculate the electric and magnetic fields due to a point charge moving in a circle of radius a at constant angular velocity ω . The charge's position and velocity are given by

$$\mathbf{w}(t) = a\hat{\mathbf{x}}\cos\omega t + a\hat{\mathbf{y}}\sin\omega t \quad (1)$$

$$\mathbf{v}(t) = -a\omega\hat{\mathbf{x}}\sin\omega t + a\omega\hat{\mathbf{y}}\cos\omega t \quad (2)$$

The calculation is easier if we use cylindrical coordinates, where

$$\hat{\mathbf{s}}(t) = \hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t \quad (3)$$

$$\hat{\phi}(t) = -\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t \quad (4)$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}} \quad (5)$$

Here, s is the radial coordinate and ϕ is the angular coordinate, and their unit vectors are both functions of time. Using these coordinates, we have

$$\mathbf{w} = a\hat{\mathbf{s}} \quad (6)$$

$$\mathbf{v} = a\omega\hat{\phi} \quad (7)$$

The fields of a moving point charge are

$$\mathbf{E}(\mathbf{r}, t) = \frac{q\mathbf{r}}{4\pi\epsilon_0(\mathbf{r}\cdot\mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a})] \quad (8)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\mathbf{t}} \times \mathbf{E}(\mathbf{r}, t) \quad (9)$$

where

$$\mathbf{r} \equiv \mathbf{r} - \mathbf{w}(t_r) \quad (10)$$

$$\mathbf{u} \equiv c\hat{\mathbf{t}} - \mathbf{v}(t_r) \quad (11)$$

$$\mathbf{a} = \dot{\mathbf{v}}(t_r) \quad (12)$$

For the fields at the centre of the circle $\mathbf{r} = 0$ so we get

$$\mathbf{r} = -\mathbf{w}(t_r) \quad (13)$$

$$= -a\hat{\mathbf{s}}(t_r) \quad (14)$$

$$\mathbf{u} = -c\hat{\mathbf{s}} - a\omega\hat{\phi} \quad (15)$$

$$\mathbf{a} = -a\omega^2\hat{\mathbf{s}} \quad (16)$$

$$\mathbf{u} \times \mathbf{a} = -a^2\omega^3\hat{\mathbf{z}} \quad (17)$$

$$\mathbf{r} \times (\mathbf{u} \times \mathbf{a}) = -a^3\omega^3\hat{\phi} \quad (18)$$

$$(c^2 - v^2)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \mathbf{a}) = (c^2 - v^2)(-c\hat{\mathbf{s}} - a\omega\hat{\phi}) - a^3\omega^3\hat{\phi} \quad (19)$$

$$= (ca^2\omega^2 - c^3)\hat{\mathbf{s}} - c^2a\omega\hat{\phi} \quad (20)$$

$$\mathbf{r} \cdot \mathbf{u} = ac \quad (21)$$

Putting all this together, we get

$$\mathbf{E}(0, t) = \frac{q}{4\pi\epsilon_0 a^2 c^2} ((a^2\omega^2 - c^2)\hat{\mathbf{s}}(t_r) - ca\omega\hat{\phi}(t_r)) \quad (22)$$

$$\mathbf{B}(0, t) = \frac{q}{4\pi\epsilon_0 a^3 c^3} (-a\hat{\mathbf{s}}) \times ((a^2\omega^2 - c^2)\hat{\mathbf{s}} - ca\omega\hat{\phi}) \quad (23)$$

$$= \frac{q\omega}{4\pi\epsilon_0 a c^2} \hat{\mathbf{z}} \quad (24)$$

$$= \frac{\mu_0 q \omega}{4\pi a} \hat{\mathbf{z}} \quad (25)$$

The electric field depends on time, but the magnetic field is constant.

We can use this result to find the magnetic field at the centre due to a steady current caused by a line charge density of λ moving at angular velocity ω . In that case each infinitesimal segment of the loop has length $a d\phi$ so $q = \lambda a d\phi$. The current is $I = \lambda a \omega$ so the field is

$$\mathbf{B} = \int_0^{2\pi} \frac{\mu_0 \lambda a \omega d\phi}{4\pi a} \hat{\mathbf{z}} \quad (26)$$

$$= \frac{\mu_0 I}{2a} \hat{\mathbf{z}} \quad (27)$$

In his example 5.6 Griffiths gives the magnetic field along the z axis of a uniform circular current, and his result is

$$\mathbf{B} = \frac{\mu_0 a^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{z}} \quad (28)$$

which reduces to our result at the centre of the circle where $z = 0$.