FIELDS OF A POINT CHARGE MOVING IN A CIRCLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.20.

We can now calculate the electric and magnetic fields due to a point charge moving in a circle of radius a at constant angular velocity ω. The charge's position and velocity are given by

$$\mathbf{w}(t) = a\hat{\mathbf{x}}\cos\omega t + a\hat{\mathbf{y}}\sin\omega t \tag{1}$$

$$\mathbf{v}(t) = -a\omega\hat{\mathbf{x}}\sin\omega t + a\omega\hat{\mathbf{y}}\cos\omega t \tag{2}$$

The calculation is easier if we use cylindrical coordinates, where

$$\hat{\mathbf{s}}(t) = \hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t \tag{3}$$

$$\hat{\phi}(t) = -\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t \tag{4}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{5}$$

Here, s is the radial coordinate and ϕ is the angular coordinate, and their unit vectors are both functions of time. Using these coordinates, we have

$$\mathbf{w} = a\hat{\mathbf{s}} \tag{6}$$

$$\mathbf{v} = a\omega\hat{\phi} \tag{7}$$

The fields of a moving point charge are

$$\mathbf{E}(\mathbf{r},t) = \frac{q\mathfrak{r}}{4\pi\varepsilon_0(\boldsymbol{\mathfrak{r}}\cdot\mathbf{u})^3}\left[\left(c^2-v^2\right)\mathbf{u}+\boldsymbol{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a})\right] \tag{8}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathfrak{r}}}\times\mathbf{E}(\mathbf{r},t) \tag{9}$$

where

$$\mathfrak{r} \equiv \mathbf{r} - \mathbf{w}(t_r) \tag{10}$$
$$\mathbf{u} \equiv c\hat{\mathfrak{r}} - \mathbf{v}(t_r) \tag{11}$$
$$\mathbf{a} = \dot{\mathbf{v}}(t_r) \tag{12}$$

For the fields at the centre of the circle $\mathbf{r} = 0$ so we get

$$\mathfrak{r} = -\mathbf{w}(t_r) \tag{13}$$
$$= -a\hat{\mathbf{s}}(t_r) \tag{14}$$
$$\mathbf{u} = -c\hat{\mathbf{s}} - a\omega\hat{\boldsymbol{\phi}} \tag{15}$$
$$\mathbf{a} = -a\omega^2\hat{\mathbf{s}} \tag{16}$$
$$\mathbf{u} \times \mathbf{a} = -a^2\omega^3\hat{\mathbf{z}} \tag{17}$$
$$\mathfrak{r} \times (\mathbf{u}\times\mathbf{a}) = -a^3\omega^3\hat{\boldsymbol{\phi}} \tag{18}$$
$$\left(c^2 - v^2\right)\mathbf{u} + \mathfrak{r} \times (\mathbf{u}\times\mathbf{a}) = \left(c^2 - v^2\right)\left(-c\hat{\mathbf{s}} - a\omega\hat{\boldsymbol{\phi}}\right) - a^3\omega^3\hat{\boldsymbol{\phi}} \tag{19}$$
$$= \left(ca^2\omega^2 - c^3\right)\hat{\mathbf{s}} - c^2 a\omega\hat{\boldsymbol{\phi}} \tag{20}$$
$$\mathfrak{r} \cdot \mathbf{u} = ac \tag{21}$$

Putting all this together, we get

$$\mathbf{E}(0,t) = \frac{q}{4\pi\varepsilon_0 a^2 c^2}\left(\left(a^2\omega^2 - c^2\right)\hat{\mathbf{s}}(t_r) - ca\omega\hat{\boldsymbol{\phi}}(t_r)\right) \tag{22}$$
$$\mathbf{B}(0,t) = \frac{q}{4\pi\varepsilon_0 a^3 c^3}(-a\hat{\mathbf{s}}) \times \left(\left(a^2\omega^2 - c^2\right)\hat{\mathbf{s}} - ca\omega\hat{\boldsymbol{\phi}}\right) \tag{23}$$
$$= \frac{q\omega}{4\pi\varepsilon_0 ac^2}\hat{\mathbf{z}} \tag{24}$$
$$= \frac{\mu_0 q\omega}{4\pi a}\hat{\mathbf{z}} \tag{25}$$

The electric field depends on time, but the magnetic field is constant.

We can use this result to find the magnetic field at the centre due to a steady current caused by a line charge density of λ moving at angular velocity ω. In that case each infinitesimal segment of the loop has length $a\,d\phi$ so $q = \lambda a\,d\phi$. The current is $I = \lambda a\omega$ so the field is

$$\mathbf{B} = \int_0^{2\pi} \frac{\mu_0 \lambda a\omega\,d\phi}{4\pi a}\hat{\mathbf{z}} \tag{26}$$
$$= \frac{\mu_0 I}{2a}\hat{\mathbf{z}} \tag{27}$$

In his example 5.6 Griffiths gives the magnetic field along the z axis of a uniform circular current, and his result is

$$\mathbf{B}=\frac{\mu_0 a^2}{2\left(a^2+z^2\right)^{3/2}}\hat{\mathbf{z}} \tag{28}$$

which reduces to our result at the centre of the circle where $z=0$.