RETARDED POTENTIALS FOR A SINUSOIDAL CURRENT LOOP

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.21.

Here's another example of calculating retarded potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{2}$$

We have a circular loop of radius a with a charge distribution (at $t=0$) of $\lambda_0\sin\frac{\theta}{2}$. The loop is then spun at angular velocity ω and we wish to find the potentials at the centre of the loop.

In this case, the distance from the observation point ($\mathbf{r}=0$) to the retarded position of the charge is always the same ($d=a$) so the potentials become

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0 d}\int\rho\left(\mathbf{r}',t_r\right)d^3\mathbf{r}' \tag{3}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi d}\int\mathbf{J}\left(\mathbf{r}',t_r\right)d^3\mathbf{r}' \tag{4}$$

The scalar potential's integral is thus just the total charge on the loop, which doesn't depend on time so we have

$$V(\mathbf{r},t) = \frac{\lambda_0 a}{4\pi\varepsilon_0 a}\int_0^{2\pi}\sin\frac{\theta}{2}d\theta \tag{5}$$

$$= \frac{\lambda_0}{\pi\varepsilon_0} \tag{6}$$

For the vector potential we need $\mathbf{J}(\mathbf{r}',t_r)$. To get this, we use the following argument. At $t=0$, the charge element at angle θ is $\lambda_0 a\sin\frac{\theta}{2}$. At time t, this element has moved through angle ωt so it's now at an angle of $\theta+\omega t$, so the linear charge density as a function of time is

$$\lambda(t) = \lambda_0 a \sin\frac{\theta}{2}\left[\cos(\theta+\omega t)\hat{\mathbf{x}} + \sin(\theta+\omega t)\hat{\mathbf{y}}\right] \tag{7}$$

The linear current is therefore

$$\mathbf{I}(\theta,t) = \dot{\lambda} = \lambda_0 a\omega \sin\frac{\theta}{2}\left[-\sin(\theta+\omega t)\hat{\mathbf{x}} + \cos(\theta+\omega t)\hat{\mathbf{y}}\right] \tag{8}$$

The integral of the current is

(9)

$$\int \mathbf{J}\left(\mathbf{r}',t_r\right) d^3\mathbf{r}' = \int_0^{2\pi} \mathbf{I}(\theta,t_r)$$

$$= \lambda_0 a\omega \int_0^{2\pi}\left[-\sin\frac{\theta}{2}\sin(\theta+\omega t_r)\hat{\mathbf{x}} + \sin\frac{\theta}{2}\cos(\theta+\omega t_r)\hat{\mathbf{y}}\right] d\theta \tag{10}$$

$$= \frac{4}{3}\lambda_0 a\omega\left[\sin\left(\omega\left(t-\frac{a}{c}\right)\right)\hat{\mathbf{x}} - \cos\left(\omega\left(t-\frac{a}{c}\right)\right)\hat{\mathbf{y}}\right] \tag{11}$$

[The integrals can be done using the trigonometric addition formulas for $\cos(x+y)$ and $\sin(x+y)$.]

The vector potential is therefore

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0\lambda_0 a\omega}{3\pi}\left[\sin\left(\omega\left(t-\frac{a}{c}\right)\right)\hat{\mathbf{x}} - \cos\left(\omega\left(t-\frac{a}{c}\right)\right)\hat{\mathbf{y}}\right] \tag{12}$$