

RETARDED POTENTIALS FOR A SINUSOIDAL CURRENT LOOP

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.21.

Here's another example of calculating retarded potentials

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{d} d^3\mathbf{r}' \quad (1)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{d} d^3\mathbf{r}' \quad (2)$$

We have a circular loop of radius a with a charge distribution (at $t = 0$) of $\lambda_0 \sin \frac{\theta}{2}$. The loop is then spun at angular velocity ω and we wish to find the potentials at the centre of the loop.

In this case, the distance from the observation point ($\mathbf{r} = 0$) to the retarded position of the charge is always the same ($d = a$) so the potentials become

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 a} \int \rho(\mathbf{r}', t_r) d^3\mathbf{r}' \quad (3)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi a} \int \mathbf{J}(\mathbf{r}', t_r) d^3\mathbf{r}' \quad (4)$$

The scalar potential's integral is thus just the total charge on the loop, which doesn't depend on time so we have

$$V(\mathbf{r}, t) = \frac{\lambda_0 a}{4\pi\epsilon_0 a} \int_0^{2\pi} \sin \frac{\theta}{2} d\theta \quad (5)$$

$$= \frac{\lambda_0}{\pi\epsilon_0} \quad (6)$$

For the vector potential we need $\mathbf{J}(\mathbf{r}', t_r)$. To get this, we use the following argument. At $t = 0$, the charge element at angle θ is $\lambda_0 a \sin \frac{\theta}{2}$. At time t , this element has moved through angle ωt so it's now at an angle of $\theta + \omega t$, so the linear charge density as a function of time is

$$\lambda(t) = \lambda_0 a \sin \frac{\theta}{2} [\cos(\theta + \omega t) \hat{\mathbf{x}} + \sin(\theta + \omega t) \hat{\mathbf{y}}] \quad (7)$$

The linear current is therefore

$$\mathbf{I}(\theta, t) = \dot{\lambda} = \lambda_0 a \omega \sin \frac{\theta}{2} [-\sin(\theta + \omega t) \hat{\mathbf{x}} + \cos(\theta + \omega t) \hat{\mathbf{y}}] \quad (8)$$

The integral of the current is

$$\int \mathbf{J}(\mathbf{r}', t_r) d^3 \mathbf{r}' = \int_0^{2\pi} \mathbf{I}(\theta, t_r) \quad (9)$$

$$= \lambda_0 a \omega \int_0^{2\pi} \left[-\sin \frac{\theta}{2} \sin(\theta + \omega t_r) \hat{\mathbf{x}} + \sin \frac{\theta}{2} \cos(\theta + \omega t_r) \hat{\mathbf{y}} \right] d\theta \quad (10)$$

$$= \frac{4}{3} \lambda_0 a \omega \left[\sin \left(\omega \left(t - \frac{a}{c} \right) \right) \hat{\mathbf{x}} - \cos \left(\omega \left(t - \frac{a}{c} \right) \right) \hat{\mathbf{y}} \right] \quad (11)$$

[The integrals can be done using the trigonometric addition formulas for $\cos(x+y)$ and $\sin(x+y)$.]

The vector potential is therefore

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 \lambda_0 a \omega}{3\pi} \left[\sin \left(\omega \left(t - \frac{a}{c} \right) \right) \hat{\mathbf{x}} - \cos \left(\omega \left(t - \frac{a}{c} \right) \right) \hat{\mathbf{y}} \right] \quad (12)$$