FORCE OF POINT CHARGE IN A HYPERBOLIC TRAJECTORY ON A FIXED POINT CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.24.

We'll now return to the point charge q_2 moving on a hyperbolic trajectory along the x axis. Its position is given by

$$x(t)=\sqrt{b^2+c^2t^2} \tag{1}$$

Suppose there is a second charge q_1 fixed at $x=0$. What force does q_1 exert on q_2 at time t? If we look at the problem from q_2's perspective, it needs to know what influence q_1 has on the point x that it (q_2, that is) currently occupies. In order for a signal to reach x at time t, it had to leave q_1 at time $t_r=t-\frac{x}{c}$. But since q_1 is fixed at the origin, the force that q_1 exerts on any charge q_2 at the point x is always just given by Coulomb's law without any retarded time, that is

$$\begin{aligned} F_2(t) &= \frac{q_1q_2}{4\pi\epsilon_0x^2} && (2)\\ &= \frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{b^2+c^2t^2} && (3)\end{aligned}$$

The total impulse delivered to q_2 is

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty}F_2(t)\,dt && (4)\\ &= \frac{q_1q_2}{4\pi\epsilon_0b^2}\int_{-\infty}^{\infty}\frac{1}{1+\frac{c^2}{b^2}t^2}dt && (5)\\ &= \frac{q_1q_2}{4\pi\epsilon_0bc} && (6)\end{aligned}$$

To calculate the force q_2 exerts on q_1 we *do* need the retarded time, since q_2 is moving. [Incidentally, it might seem that by considering q_1 as at rest and q_2 as moving, and treating the two cases differently, we're violating the principle of relativity, but we're not. The reason is that q_2 is not in an inertial frame; the hyperbolic trajectory means that its velocity is never constant, so q_1 and q_2 are not equivalent.]

The retarded time is calculated from

$$|\mathbf{r}-\mathbf{w}(t_r)| = c(t-t_r) \tag{7}$$

Here $\mathbf{r}=0$ is the location of q_1 and $\mathbf{w}(t_r)=\sqrt{b^2+c^2t_r^2}\hat{\mathbf{x}}$ is the position of q_2. We get

$$\begin{aligned}\sqrt{b^2+c^2t_r^2} &= c(t-t_r) && (8)\\ t_r &= \frac{t}{2}-\frac{b^2}{2c^2t} && (9)\end{aligned}$$

Note that as $t\to 0$, $t_r\to -\infty$ so q_2 is not visible to q_1 before $t=0$, so the retarded potential is zero for $t<0$. The force for $t>0$ is therefore (the minus sign indicates the force is to the left if both charges are the same sign)

$$\begin{aligned}F_1(t) &= -\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{b^2+\frac{c^2}{4}\left(t-\frac{b^2}{c^2t}\right)^2} && (10)\\ &= -\frac{q_1q_2}{\pi\epsilon_0}\frac{c^2t^2}{4c^2b^2t^2+(c^2t^2-b^2)^2} && (11)\\ &= -\frac{q_1q_2}{\pi\epsilon_0}\frac{c^2t^2}{(c^2t^2+b^2)^2} && (12)\end{aligned}$$

The total impulse is found by integrating F_1 from $t=0$ to infinity, since there is no force for $t<0$.

$$\begin{aligned}I_1 &= -\frac{q_1q_2}{\pi\epsilon_0}\int_0^\infty \frac{c^2t^2}{(c^2t^2+b^2)^2}dt && (13)\\ &= -\frac{q_1q_2}{4\pi\epsilon_0 bc} && (14)\end{aligned}$$

[The integral can be done by parts, although I used Maple.] Thus the two impulses are equal and opposite.