POWER DUE TO POINT CHARGE MOVING WITH CONSTANT VELOCITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.25.

Returning to the case of a point charge moving with constant velocity $\mathbf{v}$, the fields due to the charge are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} &(1)\\ \mathbf{B}(\mathbf{r},t) &= \frac{1}{c^2}\mathbf{v}\times\mathbf{E}(\mathbf{r},t) &(2)\end{aligned}$$

where $\mathbf{R}$ is a vector from the charge's location at the present time to the observer and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$. In what follows, we'll take $\mathbf{v}$ to be in the $+x$ direction.

Let's calculate the total power passing through the plane $x=a$ when the charge is at $x=0$. The power per unit area is given by the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{3}$$

If we use cylindrical coordinates, with the x axis serving as the more usual z axis, s being the radial distance from the x axis and ϕ the angular coordinate, then $\mathbf{E}$ has components in the s and x directions, and the unit vector $\hat{\mathbf{R}}$ is given by

$$\hat{\mathbf{R}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{s}}\sin\theta \tag{4}$$

where

$$\begin{aligned}\sin\theta &= \frac{s}{\sqrt{s^2+a^2}} &(5)\\ \cos\theta &= \frac{a}{\sqrt{s^2+a^2}} &(6)\\ R^2 &= s^2+a^2 &(7)\end{aligned}$$

To calculate $\mathbf{B}$, only the s component will survive the cross-product since $\mathbf{v}$ is parallel to $\hat{\mathbf{x}}$, so we get, using $\hat{\mathbf{x}}\times\hat{\mathbf{s}}=\hat{\phi}$

$$\mathbf{B}=\frac{v}{c^2}\frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\sin\theta}{s^2+a^2}\hat{\phi} \tag{8}$$

To get the power flowing through the plane $x=a$, we need the component of $\mathbf{S}$ parallel to $\hat{\mathbf{x}}$. Since $\mathbf{B}$ is in the ϕ direction and $\hat{\mathbf{s}}\times\hat{\phi}=\hat{\mathbf{x}}$, we can get the x component of $\mathbf{S}$ by multiplying $\mathbf{B}$ by the s component of $\mathbf{E}$, which we'll call E_s. We have

$$E_s=\frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\sin\theta}{s^2+a^2} \tag{9}$$

so the x component of the Poynting vector is

$$\begin{aligned} S_x &= \frac{v}{\mu_0 c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{\left(1-v^2/c^2\right)^2}{\left(1-v^2\sin^2\theta/c^2\right)^3}\frac{\sin^2\theta}{\left(s^2+a^2\right)^2} && (10)\\ &= \frac{vq^2}{16\pi^2\epsilon_0}\frac{\left(1-v^2/c^2\right)^2}{\left(1-v^2\sin^2\theta/c^2\right)^3}\frac{\sin^2\theta}{\left(s^2+a^2\right)^2} && (11)\end{aligned}$$

using $c^2=1/\mu_0\epsilon_0$.

We therefore need to integrate this over the plane $x=a$, which we can do by expressing $\sin\theta$ in terms of s using 5. The increment of area is $s\,ds\,d\phi$ so the integral is

$$P_{x=a}=\frac{vq^2\left(1-v^2/c^2\right)^2}{16\pi^2\epsilon_0}\int_0^\infty ds\int_0^{2\pi}d\phi\frac{s^3}{\left(1-v^2s^2/\left(s^2+a^2\right)c^2\right)^3\left(s^2+a^2\right)^3} \tag{12}$$

$$\begin{aligned} &= \frac{vq^2\left(1-v^2/c^2\right)^2}{8\pi\epsilon_0}\int_0^\infty\frac{s^3\,ds}{\left(s^2\left(1-v^2/c^2\right)+a^2\right)^3} && (13)\\ &= \frac{vq^2\left(1-v^2/c^2\right)^2}{8\pi\epsilon_0}\frac{1}{4a^2\left(1-v^2/c^2\right)^2} && (14)\\ &= \frac{vq^2}{32\pi\epsilon_0 a^2} && (15)\end{aligned}$$

I did the integral using Maple, but if you want to do it by hand, you can use partial fractions, since

$$\frac{s^3}{\left(s^2\left(1-v^2/c^2\right)+a^2\right)^3}=\frac{sc^6}{\left(s^2\left(c^2-v^2\right)+a^2c^2\right)^2\left(c^2-v^2\right)}-\frac{sc^8a^2}{\left(s^2\left(c^2-v^2\right)+a^2c^2\right)^3\left(c^2-v^2\right)} \tag{16}$$