

POWER DUE TO POINT CHARGE MOVING WITH CONSTANT VELOCITY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 10.25.

Returning to the case of a point charge moving with constant velocity \mathbf{v} , the fields due to the charge are

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2} \quad (1)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c^2} \mathbf{v} \times \mathbf{E}(\mathbf{r}, t) \quad (2)$$

where \mathbf{R} is a vector from the charge's location at the present time to the observer and θ is the angle between \mathbf{R} and \mathbf{v} . In what follows, we'll take \mathbf{v} to be in the $+x$ direction.

Let's calculate the total power passing through the plane $x = a$ when the charge is at $x = 0$. The power per unit area is given by the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (3)$$

If we use cylindrical coordinates, with the x axis serving as the more usual z axis, s being the radial distance from the x axis and ϕ the angular coordinate, then \mathbf{E} has components in the s and x directions, and the unit vector $\hat{\mathbf{R}}$ is given by

$$\hat{\mathbf{R}} = \hat{\mathbf{x}} \cos \theta + \hat{\mathbf{s}} \sin \theta \quad (4)$$

where

$$\sin \theta = \frac{s}{\sqrt{s^2 + a^2}} \quad (5)$$

$$\cos \theta = \frac{a}{\sqrt{s^2 + a^2}} \quad (6)$$

$$R^2 = s^2 + a^2 \quad (7)$$

To calculate \mathbf{B} , only the s component will survive the cross-product since \mathbf{v} is parallel to $\hat{\mathbf{x}}$, so we get, using $\hat{\mathbf{x}} \times \hat{\mathbf{s}} = \hat{\phi}$

$$\mathbf{B} = \frac{v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \frac{\sin \theta}{s^2 + a^2} \hat{\phi} \quad (8)$$

To get the power flowing through the plane $x = a$, we need the component of \mathbf{S} parallel to $\hat{\mathbf{x}}$. Since \mathbf{B} is in the ϕ direction and $\hat{\mathbf{s}} \times \hat{\phi} = \hat{\mathbf{x}}$, we can get the x component of \mathbf{S} by multiplying \mathbf{B} by the s component of \mathbf{E} , which we'll call E_s . We have

$$E_s = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2 \theta / c^2)^{3/2}} \frac{\sin \theta}{s^2 + a^2} \quad (9)$$

so the x component of the Poynting vector is

$$S_x = \frac{v}{\mu_0 c^2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \frac{(1 - v^2/c^2)^2}{(1 - v^2 \sin^2 \theta / c^2)^3} \frac{\sin^2 \theta}{(s^2 + a^2)^2} \quad (10)$$

$$= \frac{vq^2}{16\pi^2\epsilon_0} \frac{(1 - v^2/c^2)^2}{(1 - v^2 \sin^2 \theta / c^2)^3} \frac{\sin^2 \theta}{(s^2 + a^2)^2} \quad (11)$$

using $c^2 = 1/\mu_0\epsilon_0$.

We therefore need to integrate this over the plane $x = a$, which we can do by expressing $\sin \theta$ in terms of s using 5. The increment of area is $s ds d\phi$ so the integral is

$$P_{x=a} = \frac{vq^2 (1 - v^2/c^2)^2}{16\pi^2\epsilon_0} \int_0^\infty ds \int_0^{2\pi} d\phi \frac{s^3}{(1 - v^2 s^2 / (s^2 + a^2) c^2)^3 (s^2 + a^2)^3} \quad (12)$$

$$= \frac{vq^2 (1 - v^2/c^2)^2}{8\pi\epsilon_0} \int_0^\infty \frac{s^3 ds}{(s^2 (1 - v^2/c^2) + a^2)^3} \quad (13)$$

$$= \frac{vq^2 (1 - v^2/c^2)^2}{8\pi\epsilon_0} \frac{1}{4a^2 (1 - v^2/c^2)^2} \quad (14)$$

$$= \frac{vq^2}{32\pi\epsilon_0 a^2} \quad (15)$$

I did the integral using Maple, but if you want to do it by hand, you can use partial fractions, since

$$\frac{s^3}{(s^2(1 - v^2/c^2) + a^2)^3} = \frac{sc^6}{(s^2(c^2 - v^2) + a^2c^2)^2(c^2 - v^2)} - \frac{sc^8a^2}{(s^2(c^2 - v^2) + a^2c^2)^3(c^2 - v^2)} \quad (16)$$