FORCE AND MOMENTUM WITH TWO MOVING POINT CHARGES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 10. Problem 10.26.

Returning to the case of a point charge q_2 moving with constant velocity $\mathbf{v}$, the fields due to the charge are

$$\begin{aligned}
\mathbf{E}_2(\mathbf{r},t) &= \frac{q_2}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\alpha/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2} &(1)\\
\mathbf{B}_2(\mathbf{r},t) &= \frac{1}{c^2}\mathbf{v}\times\mathbf{E}_2(\mathbf{r},t) &(2)
\end{aligned}$$

where $\mathbf{R}$ is a vector from the charge's location at the present time to the observer and α is the angle between $\mathbf{R}$ and $\mathbf{v}$. In what follows, we'll take $\mathbf{v}$ to be in the $+z$ direction and place the charge at the origin at $t=0$.

We now place another charge q_1 at the origin (and hold it there). The force $\mathbf{F}_{12}$ of q_1 on q_2 is given by Coulomb's law, since q_1 is at rest and its field is given by

$$\mathbf{E}_1(\mathbf{r},t) = \frac{q_2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \qquad (3)$$

so the force is

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 v^2 t^2}\hat{\mathbf{z}} \qquad (4)$$

To get the force exerted by q_2 on q_1 we can use the field 1 with $\alpha=\pi$ and $\mathbf{R}=-\mathbf{v}t$:

$$\begin{aligned}
\mathbf{F}_{21} &= q_1\mathbf{E} &(5)\\
&= -\frac{q_1 q_2\left(1-v^2/c^2\right)}{4\pi\varepsilon_0 v^2 t^2}\hat{\mathbf{z}} &(6)
\end{aligned}$$

Thus Newton's third law (action = reaction) is not obeyed here. The reason is that both charges must be experiencing other forces than the force from the other charge. Since q_1 is at rest and q_2 is in uniform motion, the net force on each charge must be zero (no acceleration), so some external force cancels the electric force on each charge.

We can calculate the total linear momentum $\mathbf{p}$ stored in the fields of the two charges (or at least the time-varying part of the momentum) from the momentum density

$$\mathfrak{p} = \varepsilon_0 \mathbf{E} \times \mathbf{B} \tag{7}$$

Since q_1 is at rest, it generates no magnetic field, so the total momentum density is

$$\mathfrak{p} = \varepsilon_0 \left(\mathbf{E}_1 + \mathbf{E}_2\right) \times \mathbf{B}_2 \tag{8}$$

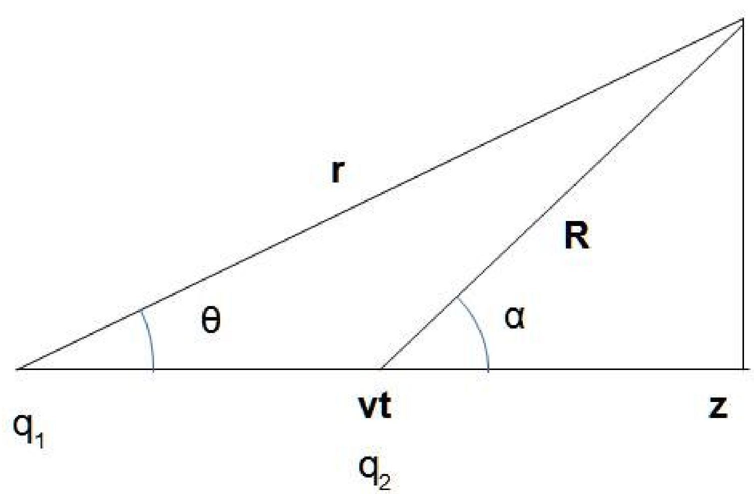
However, if we integrate $\mathbf{E}_2 \times \mathbf{B}_2$ over all space, the result (although complicated) must be independent of time since the magnetic and electric fields due to q_2 don't change relative to each other. The time varying part of the momentum density is therefore

$$\mathfrak{p}_t = \varepsilon_0 \mathbf{E}_1 \times \mathbf{B}_2 \tag{9}$$

We've already calculated $\mathbf{B}_2$ as

$$\mathbf{B}_2 = \frac{v}{c^2} \frac{q_2}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2 \alpha / c^2\right)^{3/2}} \frac{\sin \alpha}{R^2} \hat{\phi} \tag{10}$$

At this point a diagram is helpful:



We can get the direction of $\mathfrak{p}_t$ by considering the two components of $\mathbf{E}_1$. The component perpendicular to the z axis points up when we're above the z axis and down when we're below it. Since $\mathbf{B}_2$ circles around the z axis it points out of the page above the z axis and into the page below it. Since the components of both $\mathbf{E}_1$ and $\mathbf{B}_2$ reverse on either side of the z axis, their cross product is the same on both sides and points in the $+z$ direction.

The other component of $\mathbf{E}_1$ points in the $+z$ direction on both sides of the z axis so, since $\mathbf{B}_2$ reverses direction on the two sides of the z axis, the cross product of the z component of $\mathbf{E}_1$ with $\mathbf{B}_2$ at a point above the z axis is cancelled by the point on the other side of the axis. Thus the only component of $\mathbf{E}_1$ that contributes to $\mathfrak{p}_t$ when it is integrated over all space is $E_\perp$ which is, from the diagram:

$$E_\perp = \frac{q_1}{4\pi\varepsilon_0}\frac{\sin\theta}{r^2} \tag{11}$$

The momentum density is therefore

$$\begin{aligned}
\mathfrak{p}_t &= \varepsilon_0 E_\perp B_2 \hat{\mathbf{z}} & (12)\\
&= \frac{q_1 q_2 v\left(1-v^2/c^2\right)}{16\pi^2\varepsilon_0 c^2}\frac{\sin\theta}{r^2}\frac{\sin\alpha}{\left(1-v^2\sin^2\alpha/c^2\right)^{3/2}R^2}\hat{\mathbf{z}} & (13)
\end{aligned}$$

At this point, we need to choose which coordinate system to use in order to integrate this expression over all space. I originally tried cylindrical coordinates since the problem has cylindrical symmetry, but I couldn't get Maple to do the resulting integral. So we can resort to spherical coordinates if we can express α and R in terms of r and θ.

Using the cosine rule for triangles (see diagram) we have

$$\begin{aligned}
R^2 &= r^2+v^2t^2-2rvt\cos\theta & (14)\\
R\sin\alpha &= r\sin\theta & (15)\\
\sin\alpha &= \frac{r\sin\theta}{R} & (16)\\
&= \frac{r\sin\theta}{\sqrt{r^2+v^2t^2-2rvt\cos\theta}} & (17)
\end{aligned}$$

We therefore have for the total time dependent momentum

$$\mathbf{p} = \hat{\mathbf{z}}\frac{q_1 q_2 v\left(1-v^2/c^2\right)}{16\pi^2\varepsilon_0 c^2}\int_0^{\pi}\int_0^{\infty}\int_0^{2\pi}\frac{\sin\theta}{r^2}\frac{\left(r\sin\theta\right)r^2\sin\theta d\phi dr d\theta}{\left(1-\frac{v^2r^2\sin^2\theta}{c^2\left(r^2+v^2t^2-2rvt\cos\theta\right)}\right)^{3/2}\left(r^2+v^2t^2-2rvt\cos\theta\right)^{3/2}} \tag{18}$$

$$= \hat{\mathbf{z}}\frac{q_1 q_2 v\left(1-v^2/c^2\right)}{8\pi\varepsilon_0 c^2}\int_0^{\pi}\int_0^{\infty}\frac{r\sin^3\theta dr d\theta}{\left(r^2+v^2t^2-2rvt\cos\theta-\frac{1}{c^2}v^2r^2\sin^2\theta\right)^{3/2}} \tag{19}$$

$$= \hat{\mathbf{z}}\frac{q_1 q_2 v\left(1-v^2/c^2\right)}{8\pi\varepsilon_0 c^2}\int_0^{\pi}\int_0^{\infty}\frac{r\sin^3\theta dr d\theta}{\left(r^2\left(1-\frac{v^2\sin^2\theta}{c^2}\right)-2rvt\cos\theta+v^2t^2\right)^{3/2}} \tag{20}$$

$$= \hat{\mathbf{z}}\frac{q_1 q_2 v\left(1-v^2/c^2\right)}{8\pi\varepsilon_0 c^2\left(1-\frac{v^2}{c^2}\right)vt}\int_0^{\pi}\left.\frac{\sin\theta\left(r\cos\theta-vt\right)}{\sqrt{r^2\left(1-\frac{v^2\sin^2\theta}{c^2}\right)-2rvt\cos\theta+v^2t^2}}\right|_{r=0}^{r=\infty} d\theta \tag{21}$$

$$= \hat{\mathbf{z}}\frac{q_1 q_2}{8\pi\varepsilon_0 c^2 t}\int_0^{\pi}\frac{\sin\theta\left(\cos\theta+\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}\right)}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}d\theta \tag{22}$$

$$= \hat{\mathbf{z}}\frac{q_1 q_2}{8\pi\varepsilon_0 c^2 t}\left.\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}-\cos\theta\right|_0^{\pi} \tag{23}$$

$$= \hat{\mathbf{z}}\frac{\mu_0 q_1 q_2}{4\pi t} \tag{24}$$

where we used $c^2 = 1/\mu_0\varepsilon_0$ in the last line.

If we take the sum of the two forces above, we get

$$\mathbf{F}_{12}+\mathbf{F}_{21} = \frac{q_1 q_2 \mu_0}{4\pi t^2}\hat{\mathbf{z}} \tag{25}$$

$$= -\frac{d\mathbf{p}}{dt} \tag{26}$$

This indicates that the momentum change in the fields is equal and opposite to the forces between the charges, which means that the external forces are causing the momentum change. If there were no external forces, then

$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$ and there would be no change in the momentum contained in the fields.