ELECTRIC DIPOLE TIME-VARYING POTENTIALS; LORENZ GAUGE CONDITION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Section 11.1.2. Problem 11.1.

The power radiated by a system of charges and currents is defined as the amount of energy that leaves the system and is non-zero even at an infinite distance. As the total rate at which energy is radiated by electric and magnetic fields is given by the integral of the Poynting vector over a surface (e.g. a sphere) enclosing the system, then the system will radiate power if the integral over the sphere

$$P=\frac{1}{\mu_0}\int(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \tag{1}$$

is non-zero as the sphere's radius becomes infinite. For static charges and currents, both the electric and magnetic fields fall off at least as fast as $\frac{1}{r^2}$ so their product falls off at least as fast as $\frac{1}{r^4}$. Thus the integral falls off at least as fast as $\frac{1}{r^2}$ so will go to zero at infinity. Thus static charge and current distributions don't radiate. We need changing charge and/or current densities to produce radiation.

This follows from Jefimenko's equations for electric and magnetic fields

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\left(\frac{\rho(\mathbf{r}',t_r)}{d^2}\hat{\mathbf{d}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{cd}\hat{\mathbf{d}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2d}\right)d^3\mathbf{r}' \tag{2}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)\times\hat{\mathbf{d}}}{cd}+\mathbf{J}(\mathbf{r}',t_r)\times\frac{\hat{\mathbf{d}}}{d^2}\right]d^3\mathbf{r}' \tag{3}$$

where the retarded time is

$$t_r\equiv t-\frac{d}{c} \tag{4}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{5}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{6}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{7}$$

Both time-varying fields contain terms involving time derivatives $\dot{\rho}$ and $\dot{\mathbf{J}}$ that depend on $1/r$ (via their dependence on d) so the Poynting vector will have terms that depend on $1/r^2$. This means that integrating these over the surface of a sphere will cancel off the dependence on r so that the radiated energy is independent of distance from the source charges and currents.

We can see this in action for a time-varying electric dipole, which we set up as two charges connected by a wire of length l along the z axis. At the start, the top charge is $+q$ and the bottom charge is $-q$. We then drive the top charge through the wire so that it cancels out the bottom charge, then we drive the positive charge upwards again and so on, all with an angular frequency ω so that the top charge is

$$q(t) = q_0 \cos(\omega t) \tag{8}$$

To see what electric and magnetic fields this dipole produces, we need to find the scalar potential V and the vector potential $\mathbf{A}$ from which we can calculate the electric field as

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{9}$$

and the magnetic field in the usual way as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{10}$$

We can get these potentials using the formulas for retarded potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{d} d^3\mathbf{r}' \tag{11}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{d} d^3\mathbf{r}' \tag{12}$$

The details are given by Griffiths in his section 11.1.2, but the essential points to note are as follows. First, we want the fields for a perfect dipole, which is when the separation l becomes infinitesimal. This is implemented by taking $l \ll r$, that is, the distance between the charges is much less than the distance from the dipole to the observation point (ultimately we want $r \to \infty$ so this is a reasonable assumption).

Second, we assume that $l \ll c/\omega$. If the dipole gives rise to electromagnetic waves travelling at c then the wavelength of a wave with frequency ω is $\lambda = 2\pi c/\omega$ so this is equivalent to assuming that the distance between the charges is much less than the wavelength of radiation they produce. Electromagnetic waves can have very small wavelengths (visible light is in the region of 10^{-7} m) but since we're taking l to be infinitesimal, we can again say this is reasonable.

Using just these two assumptions, we can calculate the potentials by saving up to first order terms in 'small' quantities, and we get

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)+\frac{1}{r}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \quad (13)$$

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\hat{\mathbf{z}}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) \quad (14)$$

where θ is the polar angle between the z axis (the axis of the dipole) and the observation point $\mathbf{r}$ and p_0 is the maximum dipole moment $p_0 = ql$.

V can be further approximated by assuming $r \gg c/\omega$, that is, assuming that the observation point is much greater than the wavelength of the waves. Locations satisfying this condition are said to lie in the radiation zone. In this case we can drop the second term in 13 and get

$$V(r,\theta,t) = -\frac{p_0 \omega \cos\theta}{4\pi\varepsilon_0 rc}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) \quad (15)$$

We can show that the potentials 13 and 14 satisfy the Lorenz gauge condition

$$\nabla\cdot\mathbf{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t} \quad (16)$$

Since

$$r = \sqrt{x^2+y^2+z^2} \quad (17)$$

we have

$$\nabla\cdot\mathbf{A} = -\frac{\mu_0 p_0\omega}{4\pi}\frac{\partial}{\partial z}\left[\frac{1}{r}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{18}$$

$$= -\frac{\mu_0 p_0\omega}{4\pi}\left[-\frac{z}{r^3}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)+\frac{1}{r}\left(-\frac{\omega z}{rc}\right)\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{19}$$

From the geometry of the setup we have

$$\frac{z}{r} = \cos\theta \tag{20}$$

so we get

$$\nabla\cdot\mathbf{A} = \frac{\mu_0 p_0\omega\cos\theta}{4\pi r}\left[\frac{1}{r}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)+\frac{\omega}{c}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{21}$$

From 13 we have

$$-\mu_0\varepsilon_0\frac{\partial V}{\partial t} = -\frac{p_0\mu_0\cos\theta}{4\pi r}\left[-\frac{\omega^2}{c}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)-\frac{\omega}{r}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{22}$$

$$= \frac{\mu_0 p_0\omega\cos\theta}{4\pi r}\left[\frac{\omega}{c}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)+\frac{1}{r}\sin\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{23}$$

$$= \nabla\cdot\mathbf{A} \tag{24}$$

Thus the gauge condition is satisfied.

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