RADIATION FROM A ROTATING DIPOLE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 4

We can simulate a rotating dipole by superimposing two perpendicular oscillating dipoles. If our rotating dipole is located at the origin and rotates about the z axis (so the axis of the dipole lies in the xy plane), then we get

$$\mathbf{p}=p_0\left(\cos\omega t\hat{\mathbf{x}}+\sin\omega t\hat{\mathbf{y}}\right) \tag{1}$$

Since the fields obey the superposition principle (fields from 2 sources just add), we can work through the formulas we found earlier to get the fields and thus the radiated power. To simplify the notation, we'll use the shorthand

$$c_\omega \equiv \cos\left(\omega\left(t-\frac{r}{c}\right)\right) \tag{2}$$

$$c_\theta \equiv \cos\theta \tag{3}$$

$$c_\phi \equiv \cos\phi \tag{4}$$

with analogous notation for the sines of these quantities.

The fields for a dipole pointing in an arbitrary direction are

$$\mathbf{E} = -\frac{\mu_0\omega^2}{4\pi r}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\left(\hat{\mathbf{p}}_0\times\hat{\mathbf{r}}\right)\times\hat{\mathbf{r}} \tag{5}$$

$$\mathbf{B} = -\frac{\mu_0\omega^2}{4\pi rc}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\left(\hat{\mathbf{p}}_0\times\hat{\mathbf{r}}\right) \tag{6}$$

Superposing the two perpendicular dipoles we get

$$\mathbf{E} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\left[\left(c_\omega\hat{\mathbf{x}}+s_\omega\hat{\mathbf{y}}\right)\times\hat{\mathbf{r}}\right]\times\hat{\mathbf{r}} \tag{7}$$

$$\mathbf{B} = -\frac{\mu_0 p_0\omega^2}{4\pi rc}\left[\left(c_\omega\hat{\mathbf{x}}+s_\omega\hat{\mathbf{y}}\right)\times\hat{\mathbf{r}}\right] \tag{8}$$

To do the cross products we convert the rectangular unit vectors to spherical unit vectors:

$$\begin{aligned}\hat{\mathbf{x}} &= s_\theta c_\phi \hat{\mathbf{r}} + c_\theta c_\phi \hat{\theta} - s_\phi \hat{\phi} &(9)\\ \hat{\mathbf{y}} &= s_\theta s_\phi \hat{\mathbf{r}} + c_\theta s_\phi \hat{\theta} + c_\phi \hat{\phi} &(10)\end{aligned}$$

Then

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{r}} &= -c_\theta c_\phi \hat{\phi} - s_\phi \hat{\theta} &(11)\\ (\hat{\mathbf{x}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} &= -c_\theta c_\phi \hat{\theta} + s_\phi \hat{\phi} &(12)\\ \hat{\mathbf{y}} \times \hat{\mathbf{r}} &= -c_\theta s_\phi \hat{\phi} + c_\phi \hat{\theta} &(13)\\ (\hat{\mathbf{y}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} &= -c_\theta s_\phi \hat{\theta} - c_\phi \hat{\phi} &(14)\end{aligned}$$

Plugging everything in and collecting terms we get

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\left[\hat{\theta} c_\theta \left(-c_\omega c_\phi - s_\omega s_\phi\right) + \hat{\phi}\left(c_\omega s_\phi - s_\omega c_\phi\right)\right] \tag{15}$$

$$\mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c}\left[-\hat{\theta}\left(c_\omega s_\phi - s_\omega c_\phi\right) + \hat{\phi} c_\theta \left(-c_\omega c_\phi - s_\omega s_\phi\right)\right] \tag{16}$$

Defining

$$\begin{aligned}E_\theta &\equiv -c_\theta\left(c_\omega c_\phi + s_\omega s_\phi\right) &(17)\\ E_\phi &\equiv \left(c_\omega s_\phi - s_\omega c_\phi\right) &(18)\end{aligned}$$

we can write the fields as

$$\begin{aligned}\mathbf{E} &= -\frac{\mu_0 p_0 \omega^2}{4\pi r}\left[\hat{\theta} E_\theta + \hat{\phi} E_\phi\right] &(19)\\ \mathbf{B} &= -\frac{\mu_0 p_0 \omega^2}{4\pi r c}\left[-\hat{\theta} E_\phi + \hat{\phi} E_\theta\right] &(20)\end{aligned}$$

In this form, it's obvious that $\mathbf{E} \cdot \mathbf{B} = 0$, so $\mathbf{E}$ and $\mathbf{B}$ are perpendicular. The Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} &(21)\\ &= \frac{\mu_0}{c}\left(\frac{p_0 \omega^2}{4\pi r}\right)^2 \left(E_\theta^2 + E_\phi^2\right)\hat{\mathbf{r}} &(22)\\ &= \frac{\mu_0}{c}\left(\frac{p_0 \omega^2}{4\pi r}\right)^2 \left(c_\theta^2 \left(c_\omega c_\phi + s_\omega s_\phi\right)^2 + \left(c_\omega s_\phi - s_\omega c_\phi\right)^2\right)\hat{\mathbf{r}} &(23)\end{aligned}$$

The average energy radiated is the average of $\mathbf{S}$ over a single time cycle, so it's the average over the terms involving c_ω and s_ω. These are of two types: terms involving c_ω^2 or s_ω^2 and the cross terms involving $s_\omega c_\omega$. The average of $s_\omega c_\omega$ over a cycle is zero and the average of c_ω^2 or s_ω^2 is $\frac{1}{2}$, so the cross terms contribute nothing and we get

$$\langle\mathbf{S}\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2\left[\frac{1}{2}c_\theta^2\left(c_\phi^2+s_\phi^2\right)+\frac{1}{2}\left(c_\phi^2+s_\phi^2\right)\right] \tag{24}$$

$$= \frac{\mu_0}{2c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2\left(1+\cos^2\theta\right) \tag{25}$$

The average radiated power is maximum in the $\pm z$ directions where $\theta = 0,\pi$ and minimum (though not zero) in the xy plane, where $\theta = \frac{\pi}{2}$. There is no dependence on ϕ which is what we'd expect on average since the dipole rotates uniformly through all values of ϕ. [There is a dependence on ϕ within each cycle, since the radiated power in a given azimuthal direction depends on where the dipole is in its rotation.]

The total average radiated power is

$$\langle P\rangle = \frac{\mu_0}{2c}\left(\frac{p_0\omega^2}{4\pi}\right)^2\int_0^\pi\int_0^{2\pi}\frac{1+\cos^2\theta}{r^2}r^2\sin\theta d\phi d\theta \tag{26}$$

$$= \frac{\mu_0 p_0^2\omega^4}{6\pi c} \tag{27}$$

This is exactly twice the power from a single oscillating dipole. Although power doesn't ordinarily obey the superposition principle since it depends on the product of $\mathbf{E}$ and $\mathbf{B}$, it does here because the cross terms in 23 average out to zero over a time cycle, since the two perpendicular dipoles are $\frac{\pi}{2}$ out of phase. If they were exactly in phase, we would replace s_ω by c_ω everywhere in the calculation, and then the cross terms wouldn't average out to zero and the combined power would not be twice the individual power.

PINGBACKS

Pingback: Power radiated by a spinning ring of charge