

RADIATION FROM A ROTATING DIPOLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 4

We can simulate a rotating dipole by superimposing two perpendicular oscillating dipoles. If our rotating dipole is located at the origin and rotates about the z axis (so the axis of the dipole lies in the xy plane), then we get

$$(0.1) \quad \mathbf{p} = p_0 (\cos \omega t \hat{\mathbf{x}} + \sin \omega t \hat{\mathbf{y}})$$

Since the fields obey the superposition principle (fields from 2 sources just add), we can work through the formulas we found earlier to get the fields and thus the radiated power. To simplify the notation, we'll use the shorthand

$$(0.2) \quad c_\omega \equiv \cos \left(\omega \left(t - \frac{r}{c} \right) \right)$$

$$(0.3) \quad c_\theta \equiv \cos \theta$$

$$(0.4) \quad c_\phi \equiv \cos \phi$$

with analogous notation for the sines of these quantities.

The fields for a dipole pointing in an arbitrary direction are

$$(0.5) \quad \mathbf{E} = -\frac{\mu_0 \omega^2}{4\pi r} \cos \left(\omega \left(t - \frac{r}{c} \right) \right) (\hat{\mathbf{p}}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}$$

$$(0.6) \quad \mathbf{B} = -\frac{\mu_0 \omega^2}{4\pi r c} \cos \left(\omega \left(t - \frac{r}{c} \right) \right) (\hat{\mathbf{p}}_0 \times \hat{\mathbf{r}})$$

Superposing the two perpendicular dipoles we get

$$(0.7) \quad \mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} [(c_\omega \hat{\mathbf{x}} + s_\omega \hat{\mathbf{y}}) \times \hat{\mathbf{r}}] \times \hat{\mathbf{r}}$$

$$(0.8) \quad \mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} [(c_\omega \hat{\mathbf{x}} + s_\omega \hat{\mathbf{y}}) \times \hat{\mathbf{r}}]$$

To do the cross products we convert the rectangular unit vectors to spherical unit vectors:

$$(0.9) \quad \hat{\mathbf{x}} = s_\theta c_\phi \hat{\mathbf{r}} + c_\theta c_\phi \hat{\boldsymbol{\theta}} - s_\phi \hat{\boldsymbol{\phi}}$$

$$(0.10) \quad \hat{\mathbf{y}} = s_\theta s_\phi \hat{\mathbf{r}} + c_\theta s_\phi \hat{\boldsymbol{\theta}} + c_\phi \hat{\boldsymbol{\phi}}$$

Then

$$(0.11) \quad \hat{\mathbf{x}} \times \hat{\mathbf{r}} = -c_\theta c_\phi \hat{\boldsymbol{\phi}} - s_\phi \hat{\boldsymbol{\theta}}$$

$$(0.12) \quad (\hat{\mathbf{x}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -c_\theta c_\phi \hat{\boldsymbol{\theta}} + s_\phi \hat{\boldsymbol{\phi}}$$

$$(0.13) \quad \hat{\mathbf{y}} \times \hat{\mathbf{r}} = -c_\theta s_\phi \hat{\boldsymbol{\phi}} + c_\phi \hat{\boldsymbol{\theta}}$$

$$(0.14) \quad (\hat{\mathbf{y}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -c_\theta s_\phi \hat{\boldsymbol{\theta}} - c_\phi \hat{\boldsymbol{\phi}}$$

Plugging everything in and collecting terms we get

$$(0.15) \quad \mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} [\hat{\boldsymbol{\theta}} c_\theta (-c_\omega c_\phi - s_\omega s_\phi) + \hat{\boldsymbol{\phi}} (c_\omega s_\phi - s_\omega c_\phi)]$$

$$(0.16) \quad \mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} [-\hat{\boldsymbol{\theta}} (c_\omega s_\phi - s_\omega c_\phi) + \hat{\boldsymbol{\phi}} c_\theta (-c_\omega c_\phi - s_\omega s_\phi)]$$

Defining

$$(0.17) \quad E_\theta \equiv -c_\theta (c_\omega c_\phi + s_\omega s_\phi)$$

$$(0.18) \quad E_\phi \equiv (c_\omega s_\phi - s_\omega c_\phi)$$

we can write the fields as

$$(0.19) \quad \mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} [\hat{\boldsymbol{\theta}} E_\theta + \hat{\boldsymbol{\phi}} E_\phi]$$

$$(0.20) \quad \mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} [-\hat{\boldsymbol{\theta}} E_\phi + \hat{\boldsymbol{\phi}} E_\theta]$$

In this form, it's obvious that $\mathbf{E} \cdot \mathbf{B} = 0$, so \mathbf{E} and \mathbf{B} are perpendicular.

The Poynting vector is

$$(0.21) \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

$$(0.22) \quad = \frac{\mu_0}{c} \left(\frac{p_0 \omega^2}{4\pi r} \right)^2 (E_\theta^2 + E_\phi^2) \hat{\mathbf{r}}$$

$$(0.23) \quad = \frac{\mu_0}{c} \left(\frac{p_0 \omega^2}{4\pi r} \right)^2 \left(c_\theta^2 (c_\omega c_\phi + s_\omega s_\phi)^2 + (c_\omega s_\phi - s_\omega c_\phi)^2 \right) \hat{\mathbf{r}}$$

The average energy radiated is the average of \mathbf{S} over a single time cycle, so it's the average over the terms involving c_ω and s_ω . These are of two types: terms involving c_ω^2 or s_ω^2 and the cross terms involving $s_\omega c_\omega$. The average of $s_\omega c_\omega$ over a cycle is zero and the average of c_ω^2 or s_ω^2 is $\frac{1}{2}$, so the cross terms contribute nothing and we get

$$(0.24) \quad \langle \mathbf{S} \rangle = \frac{\mu_0}{c} \left(\frac{p_0 \omega^2}{4\pi r} \right)^2 \left[\frac{1}{2} c_\theta^2 (c_\phi^2 + s_\phi^2) + \frac{1}{2} (c_\phi^2 + s_\phi^2) \right]$$

$$(0.25) \quad = \frac{\mu_0}{2c} \left(\frac{p_0 \omega^2}{4\pi r} \right)^2 (1 + \cos^2 \theta)$$

The average radiated power is maximum in the $\pm z$ directions where $\theta = 0, \pi$ and minimum (though not zero) in the xy plane, where $\theta = \frac{\pi}{2}$. There is no dependence on ϕ which is what we'd expect on average since the dipole rotates uniformly through all values of ϕ . [There is a dependence on ϕ within each cycle, since the radiated power in a given azimuthal direction depends on where the dipole is in its rotation.]

The total average radiated power is

$$(0.26) \quad \langle P \rangle = \frac{\mu_0}{2c} \left(\frac{p_0 \omega^2}{4\pi} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{1 + \cos^2 \theta}{r^2} r^2 \sin \theta d\phi d\theta$$

$$(0.27) \quad = \frac{\mu_0 p_0^2 \omega^4}{6\pi c}$$

This is exactly twice the power from a single oscillating dipole. Although power doesn't ordinarily obey the superposition principle since it depends on the product of \mathbf{E} and \mathbf{B} , it does here because the cross terms in 0.23 average out to zero over a time cycle, since the two perpendicular dipoles are $\frac{\pi}{2}$ out of phase. If they were exactly in phase, we would replace s_ω by c_ω everywhere in the calculation, and then the cross terms wouldn't average out to zero and the combined power would not be twice the individual power.

PINGBACKS

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