FIELDS OF AN OSCILLATING MAGNETIC DIPOLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 5

We can analyze an oscillating magnetic dipole in a similar way to the electric dipole. We begin with a small circular current loop of radius b in the xy plane, centred at the origin. The current is driven to be alternating, so that

$$I(t) = I_0 \cos \omega t \tag{1}$$

The magnetic dipole moment of a current loop is

$$\mathbf{m} = I\mathbf{a} \tag{2}$$

where $\mathbf{a}$ is the vector area of the loop, which for a planar circular loop is just $\pi b^2 \hat{\mathbf{z}}$. Thus

$$\begin{aligned} \mathbf{m}(t) &= \pi b^2 I_0 \cos(\omega t)\,\hat{\mathbf{z}} && (3)\\ &\equiv m_0 \cos(\omega t)\,\hat{\mathbf{z}} && (4) \end{aligned}$$

If the loop is electrically neutral, the electric potential is $V = 0$, so we need to calculate only $\mathbf{A}$. The retarded potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos \omega (t - d/c)}{d} d\ell' \tag{5}$$

where the integral is taken around the loop and the retarded time is

$$t_r \equiv t - \frac{d}{c} \tag{6}$$

and

$$\begin{aligned} d &\equiv \left|\mathbf{r}-\mathbf{r}'\right| &(7)\\ &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} &(8)\\ \hat{\mathbf{d}} &= \frac{\mathbf{r}-\mathbf{r}'}{d} &(9)\end{aligned}$$

where $\mathbf{r}'$ is the position on the loop being integrated over.

To work out the integral, we can start by considering $\mathbf{r}$ to be some point in the xz plane. The line integral can be broken down into pairs of increments on the circle located at $(x,\pm y,0)$, that is, each pair of points is symmetric about the x axis. The increment $d\ell_-$ at $(x,-y,0)$ has components $(dx,dy,0)$ while the increment $d\ell_+$ at $(x,y,0)$ has components $(-dx,dy,0)$. Thus the x components cancel in pairs in the integral, while the y components add in pairs, so the net result of the integral around the entire circle is a vector pointing in the y direction. Since the circle is symmetric about the z axis, we can generalize this result to deduce that $\mathbf{A}$ has a direction that is always tangential to the circle, which means that, in spherical coordinates, it is in the ϕ direction and has the same magnitude at all points around the circle.

Griffiths goes through the calculation of $\mathbf{A}$ in detail in his section 11.1.3 so I won't repeat that here, other than to note that he uses the same approximations as were used with the electric dipole, namely that the radius of the loop b is much less than the observation distance r, and that b is also much smaller than the wavelength of the radiation, represented by the condition $b \ll c/\omega$. The result is

$$\mathbf{A}(r,\theta,t)=\frac{\mu_0 m_0}{4\pi}\frac{\sin\theta}{r}\left[\frac{1}{r}\cos\left[\omega\left(t-r/c\right)\right]-\frac{\omega}{c}\sin\left[\omega\left(t-r/c\right)\right]\right]\hat{\phi} \quad (10)$$

At this stage, Griffiths invokes a further approximation by assuming that $r \gg c/\omega$ (observer is much further away than the wavelength of the radiation). However, we can calculate the fields without making that approximation to see how much of an effect that approximation has. Since $V=0$ we have

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{11}$$

$$= \frac{\mu_0 m_0 \sin\theta}{4\pi r}\left\{\frac{\omega^2}{c}\cos\left[\omega\left(t-r/c\right)\right]+\frac{\omega}{r}\sin\left[\omega\left(t-r/c\right)\right]\right\}\hat{\phi} \tag{12}$$

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{13}$$

$$= \frac{\mu_0 m_0 \cos\theta}{2\pi r^2}\left[\frac{1}{r}\cos\left[\omega\left(t-r/c\right)\right]-\frac{\omega}{c}\sin\left[\omega\left(t-r/c\right)\right]\right]\hat{\mathbf{r}}+ \tag{14}$$

$$\frac{\mu_0 m_0 \sin\theta}{4\pi r}\left[\left(\frac{1}{r^2}-\frac{\omega^2}{c^2}\right)\cos\left[\omega\left(t-r/c\right)\right]-\frac{\omega}{rc}\sin\left[\omega\left(t-r/c\right)\right]\right]\hat{\theta}$$

These fields have the same form as those we worked out earlier for a spherical wave in vacuum.

The Poynting vector can be worked out and simplified using Maple to combine the trig products using double angle formulas:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{15}$$

$$= -\frac{\mu_0 m_0^2 \sin^2\theta}{32\pi^2 c^3 r^5}\left[\left(2c^2\omega^2 r-\omega^4 r^3\right)\cos\left[2\omega\left(t-r/c\right)\right]+\left(\omega c^3-2\omega^3 c r^2\right)\sin\left[2\omega\left(t-r/c\right)\right]-\omega^4 r^3\right]\hat{\mathbf{r}}- \tag{16}$$

$$\frac{\mu_0 m_0^2 \sin(2\theta)}{32\pi^2 c^3 r^5}\left[2c^2\omega^2 r\cos\left[2\omega\left(t-r/c\right)\right]+\left(\omega c^3-\omega^3 c r^2\right)\sin\left[2\omega\left(t-r/c\right)\right]\right]\hat{\theta}$$

To get the average intensity, we integrate $\mathbf{S}$ over a complete cycle, that is, for $t=0$ to $t=2\pi/\omega$ and then multiply by $\omega/2\pi$ to get the average. Integrating over one cycle causes each of the double angle trig functions to go through two complete cycles, so they all integrate to zero and the only term that is left is the $-\omega^4 r^3$ term in the radial component, so we get

$$\langle\mathbf{S}\rangle = \frac{\mu_0 m_0^2 \omega^4 \sin^2\theta}{32\pi^2 c^3 r^2}\hat{\mathbf{r}} \tag{17}$$

This value is the same as that obtained by assuming $r \gg c/\omega$ from the start (equation 11.39 in Griffiths).

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