ELECTRIC DIPOLE RADIATION FROM AN ARBITRARY SOURCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 8.

Having examined electromagnetic radiation from an oscillating electric dipole, we can now look at radiation from an arbitrary source of moving charges. The derivation of the results is rather long and Griffiths treats it in detail in his section 11.1.4 so I won't go over it all again here, except to point out the key assumptions made in the derivation.

To calculate the retarded potentials we start with

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{d}d^3\mathbf{r}' \tag{0.2}$$

where

$$t_r \equiv t-\frac{d}{c} \tag{0.3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{0.4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{0.5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{0.6}$$

The first assumption is that the overall size of the charge distribution is much smaller than the distance to the observer, so that

$$r'_{max} \ll r \tag{0.7}$$

This allows us to approximate by saving only up to first order terms in r'.

The second approximation is that r'_{max} is much less than all the terms

$$r'_{max} \ll \frac{c}{\left|\left(d^n\rho/dt^n\right)/\dot{\rho}\right|^{1/(n-1)}} \tag{0.8}$$

for $n \geq 2$. For an oscillating system, $\rho(t) = A\cos\omega t$ so

$$\frac{d^n\rho}{dt^n} = (-1)^n \omega^n \rho(t) \tag{0.9}$$

so

$$\left|\frac{1}{\dot{\rho}}\frac{d^n\rho}{dt^n}\right|^{1/(n-1)} = \omega \tag{0.10}$$

and this assumption is equivalent to $r'_{max} \ll \lambda$ that we made in analyzing the oscillating dipole. In practice, it means that we keep up to first order terms in r'.

After making these two assumptions, we arrive at approximate formulas for the potentials:

$$V(\mathbf{r},t) \cong \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t-r/c)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t-r/c)}{rc}\right] \tag{0.11}$$

$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t-r/c)}{r} \tag{0.12}$$

where $\mathbf{p}$ is the dipole moment

$$\mathbf{p} = \int \mathbf{r}'\rho\left(\mathbf{r}',t-r/c\right)d^3\mathbf{r}' \tag{0.13}$$

and Q is the total charge in the system.

By making a further assumption that r itself is very large (essentially approaching infinity, since ultimately we are interested only in radiation that makes it to infinity), we arrive at approximate formulas for the fields:

$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\right)\right] \tag{0.14}$$

$$\mathbf{B}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi rc}\left(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\right) \tag{0.15}$$

where in both cases $\ddot{\mathbf{p}}$ is evaluated at the retarded time $t-r/c$. Note that the fields depend on the second time derivative of the dipole moment, which means that no radiation is produced unless the charges are accelerating. The only way to accelerate something is, of course, to apply a force to it so we are doing work on the system, and this work is being converted (at least partly) into radiation.

If we use spherical coordinates with the z axis in the direction of $\ddot{\mathbf{p}}$ then the fields can be written as

$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0 \ddot{p}(t-r/c)}{4\pi r}\sin\theta\,\hat{\theta} \tag{0.16}$$

$$\mathbf{B}(\mathbf{r},t) \cong \frac{\mu_0 \ddot{p}(t-r/c)}{4\pi rc}\sin\theta\,\hat{\phi} \tag{0.17}$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \cong \frac{\mu_0 \ddot{p}^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \tag{0.18}$$

and the total radiated power is

$$P = \int \mathbf{S}\cdot d\mathbf{a} \cong \frac{\mu_0 \ddot{p}^2}{6\pi c} \tag{0.19}$$

Example. We can apply these formulas to the case of the rotating dipole. In that case, we had a dipole rotating in the xy plane, so its dipole moment is given by

$$\mathbf{p}(t-r/c) = p_0\left(\cos\omega(t-r/c)\,\hat{\mathbf{x}} + \sin\omega(t-r/c)\,\hat{\mathbf{y}}\right) \tag{0.20}$$

Therefore

$$\ddot{\mathbf{p}} = -p_0\omega^2\left(\cos\omega(t-r/c)\,\hat{\mathbf{x}} + \sin\omega(t-r/c)\,\hat{\mathbf{y}}\right) \tag{0.21}$$

so from 0.14 and 0.15 we get

(0.22)
$$\mathbf{E}(\mathbf{r},t) \cong -\frac{\mu_0 p_0\omega^2}{4\pi r}\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\left(\cos\omega(t-r/c)\,\hat{\mathbf{x}} + \sin\omega(t-r/c)\,\hat{\mathbf{y}}\right)\right)\right]$$

(0.23)
$$\mathbf{B}(\mathbf{r},t) \cong \frac{\mu_0 p_0\omega^2}{4\pi rc}\left(\hat{\mathbf{r}}\times\left(\cos\omega(t-r/c)\,\hat{\mathbf{x}} + \sin\omega(t-r/c)\,\hat{\mathbf{y}}\right)\right)$$

which are the same equations we got earlier (after swapping the orders of the cross products):

(0.24)
$$\mathbf{E}=-\frac{\mu_0 p_0\omega^2}{4\pi r}\left[\left(\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\mathbf{x}}+\sin\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\mathbf{y}}\right)\times\hat{\mathbf{r}}\right]\times\hat{\mathbf{r}}$$

(0.25)
$$\mathbf{B}=-\frac{\mu_0 p_0\omega^2}{4\pi rc}\left[\left(\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\mathbf{x}}+\sin\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\mathbf{y}}\right)\times\hat{\mathbf{r}}\right]$$

In this case, it's not convenient to use the spherical coordinate forms for the fields, since the direction of the dipole moment (and hence its second derivative) is changing with time. However, since the power 0.19 is obtained by integrating over all angles, it does give the same result, since

$$\ddot{p}^2 = p_0^2\omega^4 \tag{0.26}$$

$$P = \frac{\mu_0 p_0^2\omega^4}{6\pi c} \tag{0.27}$$

which is the same as we got earlier.

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