# ELECTRIC QUADRUPOLE RADIATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 11.

In analyzing radiation from an arbitrary configuration of charge, we made the assumption that the maximum dimension of the source is much smaller than the observation distance, so that we can retain only first order terms in $r'$, the variable that is integrated over the source. In some cases, the first order contribution is zero and in that case, we need to look at the next order. This leads to *electric quadrupole* (and magnetic dipole, but we'll leave that for now) radiation. A simple model that illustrates this is as follows.

Suppose we have two oscillating electric dipoles situated on the $z$ axis, with $\mathbf{p}_+$ at $z=+\frac{d}{2}$ and $\mathbf{p}_-$ at $z=-\frac{d}{2}$. The dipole oscillate exactly $\pi$ out of phase, so that the dipole moment of the upper dipole is always the negative of the dipole moment of the lower one. We can work out the fields of this setup by using the same approximations we used in deriving the ordinary oscillating dipole. First, we need to define a few terms. (I'd draw a diagram, but that's a painful process, so bear with me.)

Let the observation point $\mathbf{r}$ make an angle $\theta$ with the $z$ axis, and let the vector from $\mathbf{p}_+$ to $\mathbf{r}$ be $\mathbf{r}_+$ and the vector from $\mathbf{p}_-$ to $\mathbf{r}$ be $\mathbf{r}_-$. The vectors $\mathbf{r}_\pm$ make angles $\theta_\pm$ with the $z$ axis.

The potential formulas for a dipole at the origin are

$$V(r,\theta,t) = -\frac{p_0\omega\cos\theta}{4\pi\varepsilon_0 rc}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) \tag{0.1}$$

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\hat{\mathbf{z}}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) \tag{0.2}$$

However, here, the dipoles are not at the origin so we need to adapt these formulas. For $\mathbf{p}_+$ we must use $\mathbf{r}_+$ and $\theta_+$ so we have

$$V_+ = -\frac{p_0\omega\cos\theta_+}{4\pi\varepsilon_0 r_+ c}\sin\left(\omega\left(t-\frac{r_+}{c}\right)\right) \tag{0.3}$$

From the law of cosines we have

$$r_+ = \sqrt{r^2+\frac{d^2}{4}-2r\frac{d}{2}\cos\theta} \tag{0.4}$$

and from the geometry of the setup

$$r\cos\theta = r_+\cos\theta_+ + \frac{d}{2} \tag{0.5}$$

Now assuming $d \ll r$ we have

$$\begin{aligned}
r_+ &\approx r\left(1-\frac{d}{2r}\cos\theta\right) && (0.6)\\
r\cos\theta &\approx r\left(1-\frac{d}{2r}\cos\theta\right)\cos\theta_+ + \frac{d}{2} && (0.7)\\
\cos\theta_+ &\approx \frac{r\cos\theta-\frac{d}{2}}{r-\frac{d}{2}\cos\theta} && (0.8)\\
&\approx \frac{1}{r}\left(r\cos\theta-\frac{d}{2}\right)\left(1+\frac{d}{2r}\cos\theta\right) && (0.9)\\
&\approx \cos\theta+\frac{d}{2r}\left(\cos^2\theta-1\right) && (0.10)\\
&= \cos\theta-\frac{d}{2r}\sin^2\theta && (0.11)
\end{aligned}$$

Also,

$$\begin{aligned}
\sin\left(\omega\left(t-\frac{r_+}{c}\right)\right) &\approx \sin\left[\omega\left(t-\frac{r}{c}\right)+\frac{\omega d}{2c}\cos\theta\right] && (0.12)\\
&\approx \sin\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\omega d}{2c}\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] && (0.13)
\end{aligned}$$

to first order in $d$.

Plugging these into 0.3 we get

(0.14)

$$V_+ = -\frac{p_0\omega}{4\pi\varepsilon_0 cr}\left(\cos\theta-\frac{d}{2r}\sin^2\theta\right)\left(1+\frac{d}{2r}\cos\theta\right)\times \left\{\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\omega d}{2c}\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}$$

(0.15)

$$\approx -\frac{p_0\omega}{4\pi\varepsilon_0 cr}\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{d}{2r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\cos 2\theta+\frac{\omega d}{2c}\cos^2\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right]$$

Under our approximation of $d \ll r$ we can drop the middle term to get

(0.16)
$$V_{+}\approx-\frac{p_{0}\omega}{4\pi\varepsilon_{0}cr}\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\omega d}{2c}\cos^{2}\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right]$$

For $\mathbf{p}_-$ we can do the same calculation to get (note the opposite sign of $p_0$ since the dipole is opposite to the top one)

$$V_{-}=\frac{p_{0}\omega\cos\theta_{-}}{4\pi\varepsilon_{0}r_{+-}c}\sin\left(\omega\left(t-\frac{r_{-}}{c}\right)\right) \tag{0.17}$$

$$r_{-}\approx r\left(1+\frac{d}{2r}\cos\theta\right) \tag{0.18}$$

$$\cos\theta_{-}\approx\cos\theta-\frac{d}{2r}\sin^{2}\theta \tag{0.19}$$

$$\sin\left(\omega\left(t-\frac{r_{-}}{c}\right)\right)\approx\sin\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega d}{2c}\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \tag{0.20}$$

Putting this together, we get

(0.21)
$$V_{-}\approx\frac{p_{0}\omega}{4\pi\varepsilon_{0}cr}\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega d}{2c}\cos^{2}\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right]$$

The total potential is

$$\begin{aligned}
V &= V_{+}+V_{-} && (0.22)\\
&= -\frac{p_{0}\omega^{2}d}{4\pi\varepsilon_{0}c^{2}r}\cos^{2}\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] && (0.23)\\
&= -\frac{\mu_{0}p_{0}\omega^{2}d}{4\pi r}\cos^{2}\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] && (0.24)
\end{aligned}$$

using $c^2 = 1/\mu_0\varepsilon_0$.

For the vector potential, we get

$$\mathbf{A}_+ = -\frac{\mu_0 p_0 \omega}{4\pi r_+}\hat{\mathbf{z}}\sin\left(\omega\left(t-\frac{r_+}{c}\right)\right) \tag{0.25}$$

$$\approx -\frac{\mu_0 p_0 \omega}{4\pi r}\hat{\mathbf{z}}\left[\sin\left(\omega\left(t-\frac{r}{c}\right)\right)-\frac{d\omega\cos\theta}{2c}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{0.26}$$

$$\mathbf{A}_- \approx \frac{\mu_0 p_0 \omega}{4\pi r}\hat{\mathbf{z}}\left[\sin\left(\omega\left(t-\frac{r}{c}\right)\right)+\frac{d\omega\cos\theta}{2c}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right] \tag{0.27}$$

$$\mathbf{A} \approx -\frac{\mu_0 p_0 \omega^2 d}{4\pi rc}\hat{\mathbf{z}}\cos\theta\cos\left(\omega\left(t-\frac{r}{c}\right)\right) \tag{0.28}$$

With the potentials, we can calculate the fields. To simplify the notation, we'll use the shorthand

$$c_\omega \equiv \cos\left(\omega\left(t-\frac{r}{c}\right)\right) \tag{0.29}$$

$$c_\theta \equiv \cos\theta \tag{0.30}$$

and so on.

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{0.31}$$

$$= -\frac{\mu_0 d\omega^2 p_0}{4\pi r^2}c_\theta^2 c_\omega \hat{\mathbf{r}} + \frac{\mu_0 p_0 d\omega^2}{\pi}c_\theta s_\theta\left(\frac{\omega s_\omega}{4cr}-\frac{c_\omega}{2r^2}\right)\hat{\theta} \tag{0.32}$$

Using the approximation $r \gg c/\omega$ we can drop all but one term to get

$$\mathbf{E} \approx \frac{\mu_0 p_0 d\omega^3}{4\pi rc}c_\theta s_\theta s_\omega \hat{\theta} \tag{0.33}$$

For the magnetic field, we get

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{0.34}$$

$$= \frac{\mu_0 p_0 d\omega^2}{\pi rc}c_\theta s_\theta\left(\frac{\omega s_\omega}{4c}-\frac{c_\omega}{2r}\right)\hat{\phi} \tag{0.35}$$

Again, using the approximation $r \gg c/\omega$ we drop the second term to get

$$\mathbf{B} \approx -\frac{\mu_0 p_0 d\omega^3}{4\pi rc^2}c_\theta s_\theta s_\omega \hat{\phi} \tag{0.36}$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{0.37}$$

$$\frac{\mu_0}{c}\left(\frac{p_0 d\omega^3}{4\pi rc}\right)^2 \left(c_\theta s_\theta s_\omega\right)^2 \hat{\mathbf{r}} \tag{0.38}$$

The intensity is the time average of $\mathbf{S}$:

$$\langle\mathbf{S}\rangle = \frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3 r^2}\left(c_\theta s_\theta\right)^2 \hat{\mathbf{r}} \tag{0.39}$$

and the power is the integral of this over a sphere of radius $r$:

$$\langle P\rangle = \int \mathbf{S}\cdot d\mathbf{a} \tag{0.40}$$

$$= 2\pi\frac{\mu_0 p_0^2 d^2 \omega^6}{32\pi^2 c^3}\int_0^\pi \cos^2\theta\sin^3\theta d\theta \tag{0.41}$$

$$= \frac{\mu_0 p_0^2 d^2 \omega^6}{60\pi c^3} \tag{0.42}$$