LIÉNARD'S GENERALIZATION OF THE LARMOR FORMULA FOR AN ACCELERATING CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 15.

For an accelerating charge q that is instantaneously at rest, the power radiated is given by the Larmor formula:

$$P=\frac{\mu_0 q^2 a^2}{6\pi c} \tag{1}$$

where the acceleration a is a function of time. Griffiths shows in his section 11.2.1 that the Larmor formula is the integral of the Poynting vector over a large sphere. Another way of looking at it is that the charge radiates an amount of power dP into an element of solid angle $d\Omega=\sin\theta d\theta d\phi$ and the formula for this is obtained from the Poynting vector by multiplying by r^2 to make the result independent of distance from the charge.

$$\frac{dP}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\sin^2\theta \tag{2}$$

We can see from this formula that the angle at which the maximum power is radiated is $\theta_{max}=\frac{\pi}{2}$.

To generalize these formulas to the case where $v\neq 0$ requires a bit of a slog through the mathematics, but the results are quoted by Griffiths as his equations 11.72 and 11.73:

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{\mu_0 q^2}{16\pi^2 c^2}\frac{\left|\hat{\boldsymbol{\mathfrak{r}}}\times(\mathbf{u}\times\mathbf{a})\right|^2}{(\hat{\boldsymbol{\mathfrak{r}}}\cdot\mathbf{u})^5} &(3)\\ P &= \frac{\mu_0 q^2\gamma^6}{6\pi c}\left[a^2-\frac{|\mathbf{v}\times\mathbf{a}|^2}{c^2}\right] &(4)\end{aligned}$$

where $\hat{\boldsymbol{\mathfrak{r}}}$ is a unit vector pointing from the charge at the retarded time to the observation point on the enclosing sphere and $\mathbf{u}=c\hat{\boldsymbol{\mathfrak{r}}}-\mathbf{v}$. These formulas are Liénard's generalization of the Larmor formula.

For a charge with $\mathbf{v}\parallel\mathbf{a}$ (at the instant of retarded time), Griffiths shows in his Example 11.3 that

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{5}$$

where $\beta \equiv v/c$. This formula reduces to 2 when $\beta = 0$.

We can find the angle θ_{max} for the case $\beta \neq 0$ by differentiating 5 and setting the result to 0.

$$\frac{d}{d\theta}\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left[2\,\frac{\sin(\theta)\cos(\theta)}{(1-\beta\,\cos(\theta))^5} - 5\,\frac{(\sin(\theta))^3\,\beta}{(1-\beta\,\cos(\theta))^6}\right] = 0 \tag{6}$$

The solution $\theta = 0$ gives the angle of *minimum* power, so if we take $\theta \neq 0$ we can cancel off $\sin\theta$ and then multiply through by $(1-\beta\,\cos(\theta))^6$ to get

$$2\,(1-\beta\,\cos(\theta))\cos(\theta) - 5\left(1-(\cos(\theta))^2\right)\beta = 0 \tag{7}$$

which has the solution

$$\theta_{max} = \arccos\left[\frac{\sqrt{15\beta^2+1}-1}{3\beta}\right] \tag{8}$$

In the ultrarelativistic case, we can write

$$\beta = 1 - x \tag{9}$$

where $x \ll 1$ and expand in a Taylor series (using Maple to do the heavy lifting):

$$\begin{aligned}\frac{\sqrt{15\beta^2+1}-1}{3\beta} &= \frac{\sqrt{15\,(1-x)^2+1}-1}{3\,(1-x)} && (10)\\ &= 1-\frac{x}{4}-\frac{27}{128}x^2+\mathcal{O}\left(x^3\right) && (11)\end{aligned}$$

Since x is small, the series on the right is very close to 1, which means that the argument of the arccos is close to 1, so θ_{max} is close to 0, so we can approximate

$$\cos\theta_{max} \approx 1-\frac{\theta_{max}^2}{2} \quad (12)$$

$$\approx 1-\frac{x}{4} \quad (13)$$

$$\theta_{max} \approx \sqrt{\frac{x}{2}} \quad (14)$$

$$= \sqrt{\frac{1-\beta}{2}} \quad (15)$$

To compare the power output at the maximum angles in the two cases, we have from 2

$$\frac{dP}{d\Omega}_{rest} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \quad (16)$$

and from 5

$$\frac{dP}{d\Omega}_{rel} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta_{max}}{(1-\beta\cos\theta_{max})^5} \quad (17)$$

so the ratio is

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} = \frac{\sin^2\theta_{max}}{(1-\beta\cos\theta_{max})^5} \quad (18)$$

For $\theta_{max} \approx \sqrt{\frac{1-\beta}{2}} \ll 1$ we can approximate

$$\sin^2\theta_{max} \approx \frac{1-\beta}{2} \quad (19)$$

$$= \frac{x}{2} \quad (20)$$

$$1-\beta\cos\theta_{max} \approx 1-\beta\left(1-\frac{1-\beta}{4}\right) \quad (21)$$

$$\approx 1-(1-x)\left(1-\frac{x}{4}\right) \quad (22)$$

$$\approx \frac{5x}{4} \quad (23)$$

Therefore

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} \approx \frac{4^5 x}{2(5x)^5} \tag{24}$$

$$= \frac{512}{3125(1-\beta)^4} \tag{25}$$

To express this in terms of the relativistic factor $\gamma = 1/\sqrt{1-\beta^2}$ we can approximate γ for $\beta = 1-x$:

$$\gamma = \frac{1}{\sqrt{1-(1-x)^2}} \tag{26}$$

$$= \frac{1}{\sqrt{2x-x^2}} \tag{27}$$

$$\approx \frac{1}{\sqrt{2x}} \tag{28}$$

$$= \frac{1}{\sqrt{2(1-\beta)}} \tag{29}$$

Therefore

$$\frac{1}{(1-\beta)^4} \approx 16\gamma^8 \tag{30}$$

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} \approx 2.62\gamma^8 \tag{31}$$

Since γ gets very large for $\beta \approx 1$, the power generated by a relativistic charge is enormously greater than that of a charge at rest.

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