

SYNCHROTRON RADIATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 16.

One common instance of an accelerated charge is a charge moving in a circle. In this case the particle's instantaneous velocity \mathbf{v} is always perpendicular to its instantaneous acceleration \mathbf{a} . This is known as *synchrotron radiation*, since it is the radiation given off by particles in a synchrotron particle accelerator, where charged particles move in circular orbits between the poles of a magnet.

We can use the Liénard formulato work out the power radiated by such a charge:

$$(0.1) \quad \frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c^2} \frac{|\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\mathbf{t}} \cdot \mathbf{u})^5}$$

$$(0.2) \quad P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left[a^2 - \frac{|\mathbf{v} \times \mathbf{a}|^2}{c^2} \right]$$

At one instant of time, we can take

$$(0.3) \quad \mathbf{v} = v\hat{\mathbf{z}}$$

$$(0.4) \quad \mathbf{a} = a\hat{\mathbf{x}}$$

$$(0.5) \quad \hat{\mathbf{t}} = s_\theta c_\phi \hat{\mathbf{x}} + s_\theta s_\phi \hat{\mathbf{y}} + c_\theta \hat{\mathbf{z}}$$

$$(0.6) \quad \mathbf{u} = c\hat{\mathbf{t}} - \mathbf{v}$$

where we're using our usual shorthand for trig functions: $s_\theta \equiv \sin \theta$, $c_\theta \equiv \cos \theta$ and so on. We can now work out the components of 0.1:

$$(0.7) \quad \hat{\mathbf{t}} \cdot \mathbf{u} = c\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} - \nu c_\theta$$

$$(0.8) \quad = c - \nu c_\theta$$

$$(0.9) \quad = c(1 - \beta c_\theta)$$

$$(0.10) \quad \mathbf{u} \times \mathbf{a} = c\hat{\mathbf{t}} \times a\hat{\mathbf{x}} - \nu\hat{\mathbf{z}} \times a\hat{\mathbf{x}}$$

$$(0.11) \quad = ca(-s_\theta s_\phi \hat{\mathbf{z}} + c_\theta \hat{\mathbf{y}}) - av\hat{\mathbf{y}}$$

$$(0.12) \quad \hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ s_\theta c_\phi & s_\theta s_\phi & c_\theta \\ 0 & cac_\theta - av & -cas_\theta s_\phi \end{vmatrix}$$

$$(0.13) \quad = \hat{\mathbf{x}} \left[-cas_\theta^2 s_\phi^2 - c_\theta (cac_\theta - av) \right] + \hat{\mathbf{y}} cas_\theta^2 s_\phi c_\phi + \hat{\mathbf{z}} (cas_\theta c_\theta c_\phi - avs_\theta c_\phi)$$

Taking the square of this last vector leads to a lengthy expression which can be simplified by applying $s^2 + c^2 = 1$ repeatedly. We get, using $\beta = v/c$:

$$(0.14) \quad \frac{1}{a^2 c^2} |\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2 = s_\theta^4 c_\phi^4 + 1 + c_\theta^2 \beta^2 - 2s_\theta^2 c_\phi^2 + 2s_\theta^2 c_\phi^2 c_\theta \beta - 2c_\theta \beta + s_\theta^4 s_\phi^2 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 - 2s_\theta^2 c_\phi^2 c_\theta \beta + s_\theta^2 c_\phi^2 \beta^2$$

We can simplify this as follows. The first and seventh terms combine to give

$$(0.15) \quad s_\theta^4 c_\phi^4 + s_\theta^4 s_\phi^2 c_\phi^2 = s_\theta^4 c_\phi^2 (c_\phi^2 + s_\phi^2) = s_\theta^4 c_\phi^2$$

Combining this with the eighth term:

$$(0.16) \quad s_\theta^4 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 = s_\theta^2 c_\phi^2 (s_\theta^2 + c_\theta^2) = s_\theta^2 c_\phi^2$$

Combining this with the fourth and last terms we get

$$(0.17) \quad s_\theta^2 c_\phi^2 - 2s_\theta^2 c_\phi^2 + s_\theta^2 c_\phi^2 \beta^2 = -(1 - \beta^2) s_\theta^2 c_\phi^2$$

The second, third and sixth terms combine to give

$$(0.18) \quad 1 + c_\theta^2 \beta^2 - 2c_\theta \beta = (1 - \beta c_\theta)^2$$

Finally, the fifth and ninth terms cancel, so we're left with

$$(0.19) \quad \frac{1}{a^2 c^2} |\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2 = (1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2$$

Putting everything together we get

$$(0.20) \quad \frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2}{(1 - \beta c_\theta)^5}$$

To get the total power, we need to integrate this over all solid angles, so we get

$$(0.21) \quad P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^\pi \int_0^{2\pi} d\phi d\theta s_\theta \frac{(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2}{(1 - \beta c_\theta)^5}$$

The integral over ϕ is easy, using

$$(0.22) \quad \int_0^{2\pi} c_\phi^2 d\phi = \pi$$

so we're left with the integral over θ :

$$(0.23) \quad P = \frac{\mu_0 q^2 a^2}{16\pi c} \int_0^\pi d\theta \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2}{(1 - \beta c_\theta)^5} s_\theta$$

$$(0.24) \quad = \frac{\mu_0 q^2 a^2}{16\pi c} \int_0^\pi d\theta \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2) (1 - c_\theta^2)}{(1 - \beta c_\theta)^5} s_\theta$$

This nasty looking integral can be done by using partial fractions, since it is the ratio of two polynomials in c_θ . I did the integral using Maple, but if you're interested in doing it by hand, the partial fraction decomposition is

$$(0.25) \quad \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2) (1 - c_\theta^2)}{(1 - \beta c_\theta)^5} = -\frac{(\beta^4 - 2\beta^2 + 1)}{\beta^2 (\beta \cos(\theta) - 1)^5} + 2\frac{(\beta^2 - 1)}{\beta^2 (\beta \cos(\theta) - 1)^4} - \frac{(\beta^2 + 1)}{(\beta \cos(\theta) - 1)^3}$$

The presence of the extra $\sin \theta$ from the solid angle element saves the day, since it multiplies each term in the partial fraction expansion, providing the derivative of $\cos \theta$ on the top of each fraction. For example

$$(0.26) \quad \int d\theta \frac{\sin \theta}{(\beta \cos(\theta) - 1)^5} = \frac{1}{4\beta (\beta \cos(\theta) - 1)^4}$$

with the other two terms having similar integrals.

The result of the integral is

$$(0.27) \quad \int_0^\pi d\theta \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2)(1 - c_\theta^2)}{(1 - \beta c_\theta)^5} s_\theta = \frac{8}{3(1 - \beta)^2(1 + \beta)^2}$$

$$(0.28) \quad = \frac{8}{3(1 - \beta^2)^2}$$

$$(0.29) \quad = \frac{8\gamma^4}{3}$$

so we get for the total power

$$(0.30) \quad P = \frac{\mu_0 q^2 a^2}{16\pi c} \frac{8\gamma^4}{3}$$

$$(0.31) \quad = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}$$