

## SYNCHROTRON RADIATION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 16.

One common instance of an accelerated charge is a charge moving in a circle. In this case the particle's instantaneous velocity  $\mathbf{v}$  is always perpendicular to its instantaneous acceleration  $\mathbf{a}$ . This is known as *synchrotron radiation*, since it is the radiation given off by particles in a synchrotron particle accelerator, where charged particles move in circular orbits between the poles of a magnet.

We can use the Liénard formulato work out the power radiated by such a charge:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c^2} \frac{|\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\mathbf{t}} \cdot \mathbf{u})^5} \quad (1)$$

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left[ a^2 - \frac{|\mathbf{v} \times \mathbf{a}|^2}{c^2} \right] \quad (2)$$

At one instant of time, we can take

$$\mathbf{v} = v\hat{\mathbf{z}} \quad (3)$$

$$\mathbf{a} = a\hat{\mathbf{x}} \quad (4)$$

$$\hat{\mathbf{t}} = s_\theta c_\phi \hat{\mathbf{x}} + s_\theta s_\phi \hat{\mathbf{y}} + c_\theta \hat{\mathbf{z}} \quad (5)$$

$$\mathbf{u} = c\hat{\mathbf{t}} - \mathbf{v} \quad (6)$$

where we're using our usual shorthand for trig functions:  $s_\theta \equiv \sin \theta$ ,  $c_\theta \equiv \cos \theta$  and so on. We can now work out the components of 1:

$$\hat{\mathbf{t}} \cdot \mathbf{u} = c\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} - vc_\theta \quad (7)$$

$$= c - vc_\theta \quad (8)$$

$$= c(1 - \beta c_\theta) \quad (9)$$

$$\mathbf{u} \times \mathbf{a} = c\hat{\mathbf{t}} \times a\hat{\mathbf{x}} - v\hat{\mathbf{z}} \times a\hat{\mathbf{x}} \quad (10)$$

$$= ca(-s_\theta s_\phi \hat{\mathbf{z}} + c_\theta \hat{\mathbf{y}}) - av\hat{\mathbf{y}} \quad (11)$$

$$\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ s_\theta c_\phi & s_\theta s_\phi & c_\theta \\ 0 & cac_\theta - av & -cas_\theta s_\phi \end{vmatrix} \quad (12)$$

$$= \hat{\mathbf{x}} \left[ -cas_\theta^2 s_\phi^2 - c_\theta (cac_\theta - av) \right] + \quad (13)$$

$$\hat{\mathbf{y}} cas_\theta^2 s_\phi c_\phi + \hat{\mathbf{z}} (cas_\theta c_\theta c_\phi - avs_\theta c_\phi)$$

Taking the square of this last vector leads to a lengthy expression which can be simplified by applying  $s^2 + c^2 = 1$  repeatedly. We get, using  $\beta = v/c$ :

$$\frac{1}{a^2 c^2} |\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2 = s_\theta^4 c_\phi^4 + 1 + c_\theta^2 \beta^2 - 2s_\theta^2 c_\phi^2 + 2s_\theta^2 c_\phi^2 c_\theta \beta - 2c_\theta \beta + \quad (14)$$

$$s_\theta^4 s_\phi^2 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 - 2s_\theta^2 c_\phi^2 c_\theta \beta + s_\theta^2 c_\phi^2 \beta^2$$

We can simplify this as follows. The first and seventh terms combine to give

$$s_\theta^4 c_\phi^4 + s_\theta^4 s_\phi^2 c_\phi^2 = s_\theta^4 c_\phi^2 (c_\phi^2 + s_\phi^2) = s_\theta^4 c_\phi^2 \quad (15)$$

Combining this with the eighth term:

$$s_\theta^4 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 = s_\theta^2 c_\phi^2 (s_\theta^2 + c_\theta^2) = s_\theta^2 c_\phi^2 \quad (16)$$

Combining this with the fourth and last terms we get

$$s_\theta^2 c_\phi^2 - 2s_\theta^2 c_\phi^2 + s_\theta^2 c_\phi^2 \beta^2 = -(1 - \beta^2) s_\theta^2 c_\phi^2 \quad (17)$$

The second, third and sixth terms combine to give

$$1 + c_\theta^2 \beta^2 - 2c_\theta \beta = (1 - \beta c_\theta)^2 \quad (18)$$

Finally, the fifth and ninth terms cancel, so we're left with

$$\frac{1}{a^2 c^2} |\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2 = (1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2 \quad (19)$$

Putting everything together we get

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2}{(1 - \beta c_\theta)^5} \quad (20)$$

To get the total power, we need to integrate this over all solid angles, so we get

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^\pi \int_0^{2\pi} d\phi d\theta s_\theta \frac{(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2 c_\phi^2}{(1 - \beta c_\theta)^5} \quad (21)$$

The integral over  $\phi$  is easy, using

$$\int_0^{2\pi} c_\phi^2 d\phi = \pi \quad (22)$$

so we're left with the integral over  $\theta$ :

$$P = \frac{\mu_0 q^2 a^2}{16\pi c} \int_0^\pi d\theta \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2) s_\theta^2}{(1 - \beta c_\theta)^5} s_\theta \quad (23)$$

$$= \frac{\mu_0 q^2 a^2}{16\pi c} \int_0^\pi d\theta \frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2)(1 - c_\theta^2)}{(1 - \beta c_\theta)^5} s_\theta \quad (24)$$

This nasty looking integral can be done by using partial fractions, since it is the ratio of two polynomials in  $c_\theta$ . I did the integral using Maple, but if you're interested in doing it by hand, the partial fraction decomposition is

$$\frac{2(1 - \beta c_\theta)^2 - (1 - \beta^2)(1 - c_\theta^2)}{(1 - \beta c_\theta)^5} = -\frac{(\beta^4 - 2\beta^2 + 1)}{\beta^2 (\beta \cos(\theta) - 1)^5} + 2\frac{(\beta^2 - 1)}{\beta^2 (\beta \cos(\theta) - 1)^4} - \frac{(\beta^2 + 1)}{(\beta \cos(\theta) - 1)^3} \quad (25)$$

The presence of the extra  $\sin \theta$  from the solid angle element saves the day, since it multiplies each term in the partial fraction expansion, providing the derivative of  $\cos \theta$  on the top of each fraction. For example

$$\int d\theta \frac{\sin \theta}{(\beta \cos(\theta) - 1)^5} = \frac{1}{4\beta (\beta \cos(\theta) - 1)^4} \quad (26)$$

with the other two terms having similar integrals.

The result of the integral is

$$\int_0^\pi d\theta \frac{2(1-\beta c_\theta)^2 - (1-\beta^2)(1-c_\theta^2)}{(1-\beta c_\theta)^5} s_\theta = \frac{8}{3(1-\beta)^2(1+\beta)^2} \quad (27)$$

$$= \frac{8}{3(1-\beta^2)^2} \quad (28)$$

$$= \frac{8\gamma^4}{3} \quad (29)$$

so we get for the total power

$$P = \frac{\mu_0 q^2 a^2}{16\pi c} \frac{8\gamma^4}{3} \quad (30)$$

$$= \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c} \quad (31)$$