RADIATION REACTION: THE ABRAHAM-LORENTZ FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 19.

If a charged particle accelerates, it radiates away energy. This means that if an external force is applied to a charge, not all of the energy transferred to the charge by the force is converted to the kinetic energy of the charge; some of the energy is radiated away in the form of electromagnetic waves. Looked at another way, the charge accelerates less than an uncharged particle of the same mass. From Newton's law $F = ma$, the *net* force on the charge must be less than the applied external force. In effect, the fields surrounding the charge exert a 'recoil' or *reaction force* on the charge.

Since the power radiated by a charge (in the non-relativistic case, anyway) is given by the Larmor formula, we might expect that this formula could be used to work out the reaction force, $\mathbf{F}_{rad}$. If this force acts on the charge as it moves a distance $\mathbf{r}$, the work done is $\mathbf{F}_{rad} \cdot \mathbf{r}$, so the *rate* at which this force does work, which is the power lost to radiation, is the time derivative of this, or $\mathbf{F}_{rad} \cdot \mathbf{v}$. The problem with this argument is that the Larmor formula measures only that radiation that extends out to infinity. The fields of a moving point charge are (with $\mathbf{u} = \hat{\boldsymbol{\mathfrak{r}}} c - \mathbf{v}$)

$$\mathbf{E}(\mathbf{r},t) = \frac{q\mathfrak{r}}{4\pi\epsilon_0 (\boldsymbol{\mathfrak{r}} \cdot \mathbf{u})^3}\left[\left(c^2 - v^2\right)\mathbf{u} + \boldsymbol{\mathfrak{r}} \times (\mathbf{u} \times \mathbf{a})\right] \tag{1}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathfrak{r}}} \times \mathbf{E}(\mathbf{r},t) \tag{2}$$

and it is only those terms that go as $\frac{1}{\mathfrak{r}}$ that contribute to radiation that is measured by the Larmor formula. The other term, namely $q\mathfrak{r}\left(c^2 - v^2\right)\mathbf{u}/4\pi\epsilon_0 (\boldsymbol{\mathfrak{r}} \cdot \mathbf{u})^3$ (and its counterpart in $\mathbf{B}$), falls off as $1/\mathfrak{r}^2$ so contributes nothing to the integral of the Poynting vector over a large sphere, since it gets multiplied into either another term in $1/\mathfrak{r}^2$ or the term in $1/\mathfrak{r}$, giving terms in $1/\mathfrak{r}^4$ or $1/\mathfrak{r}^3$. This $1/\mathfrak{r}^2$ term is called the *velocity field* (for lack of a better name; it's a bit misleading as both the velocity field and the acceleration field depend on velocity, but never mind) and, although it doesn't contribute to the radiation in the Larmor formula, it *does* store energy, so some of the energy imparted by the force that gets the charge moving must be siphoned off to create these velocity fields.

These velocity fields are curious beasts, however, for they contain energy that is never actually lost to the charge. If a charge is accelerated to some velocity, the velocity fields are constructed around the moving charge, but if the charge is then decelerated to rest again, the velocity fields disappear without having radiated away any energy. Where does this energy go? Griffiths doesn't address this point, but it would seem to be reabsorbed by the charge as it slows down.

In any case, if we look at a charged particle that starts off in some state, then goes through an acceleration followed by a deceleration, and finally ends up in the same state that it started from, what we *can* say is that the velocity fields are the same at the end as they were at the start, so over this period, the only energy that is truly lost from the particle is the energy that is radiated away, that is, the energy measured by the Larmor formula, which states

$$P=\frac{\mu_0 q^2 a^2}{6\pi c} \tag{3}$$

Therefore, if we average the energy over such a period, we get

$$\begin{aligned}
\int_{t_1}^{t_2}\mathbf{F}_{rad}\cdot\mathbf{v}dt &= -\int_{t_1}^{t_2}Pdt \qquad (4)\\
&= -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2}a^2 dt \qquad (5)\\
\int_{t_1}^{t_2}a^2 dt &= \int_{t_1}^{t_2}\dot{\mathbf{v}}\cdot\dot{\mathbf{v}}dt \qquad (6)\\
&= \mathbf{v}\cdot\dot{\mathbf{v}}\big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\mathbf{v}\cdot\ddot{\mathbf{v}}dt \qquad (7)
\end{aligned}$$

The integrated term in the last line is zero because the charge is in the same state at t_1 and t_2 so its velocity and acceleration are the same at those two times. Combining the remaining integral with the LHS in 4 we get

$$\begin{aligned}
\int_{t_1}^{t_2}\mathbf{F}_{rad}\cdot\mathbf{v}dt &= -\frac{\mu_0 q^2}{6\pi c}\left[-\int_{t_1}^{t_2}\mathbf{v}\cdot\ddot{\mathbf{v}}dt\right] \qquad (8)\\
\int_{t_1}^{t_2}\left(\mathbf{F}_{rad}-\frac{\mu_0 q^2}{6\pi c}\ddot{\mathbf{v}}\right)\cdot\mathbf{v}dt &= 0 \qquad (9)
\end{aligned}$$

Since $\ddot{\mathbf{v}} = \dot{\mathbf{a}}$, one possible solution of this equation is if the quantity in parentheses is identically zero, or

$$\boxed{\mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}} \tag{10}$$

This is known as the *Abraham-Lorentz formula* for radiation reaction. Note that it depends on the rate of change of the acceleration, and not on the acceleration itself, so the reaction force can be non-zero even when the acceleration is momentarily zero, that is, at a time when the particle is not radiating.

This formula has become infamous in classical electrodynamics as it makes predictions which are seemingly at odds with common sense. One argument goes as follows. If $\mathbf{F}_{rad}$ is the only force acting on the particle then from Newton's law

$$F_{rad} = ma = \frac{\mu_0 q^2}{6\pi c}\dot{a} \tag{11}$$

This differential equation has the general solution

$$a(t) = a_0 e^{t/\tau} \tag{12}$$

where

$$\tau \equiv \frac{\mu_0 q^2}{6\pi mc} \tag{13}$$

From this it seems that the acceleration spontaneously increases with time. However, I'm not sure why this particular point is considered to be a problem, because if (as we assumed) the charge is subject to no *external* force, then surely its velocity must be constant (from Newton's law) so the acceleration and all its derivatives are zero, so $F_{rad} = 0$ as well. This amounts to taking $a_0 = 0$ in the solution above. Problem solved. Or not

Before we do an example indicating the problem in more detail, first consider the general case where a charge is subject to a reaction force and a separate external force F. In that case

$$ma = F_{rad} + F \tag{14}$$

$$= \frac{\mu_0 q^2}{6\pi c}\dot{a} + F \tag{15}$$

$$a = \tau\dot{a} + \frac{F}{m} \tag{16}$$

For an electrically neutral particle, the reaction term disappears, meaning that if F is discontinuous (such as a force that is switched on at a certain time), then so too is a. However, it turns out that the addition of the $\tau\dot{a}$ term guarantees that the acceleration is always continuous, even if the force is not. Suppose we integrate this equation over some small time interval from $t-\varepsilon$ to $t+\varepsilon$. Then we get

$$\int_{t-\varepsilon}^{t+\varepsilon} a\,dt = \tau\int_{t-\varepsilon}^{t+\varepsilon}\dot{a}\,dt + \frac{1}{m}\int_{t-\varepsilon}^{t+\varepsilon} F\,dt \tag{17}$$

$$= \tau\left(a\left(t+\varepsilon\right) - a\left(t-\varepsilon\right)\right) + \frac{1}{m}\int_{t-\varepsilon}^{t+\varepsilon} F\,dt \tag{18}$$

Provided both a and F are finite over the time interval, both their integrals must tend to zero as $\varepsilon \to 0$. In particular, F is allowed to be discontinuous (but it can't contain any infinities, such as a delta function). In that case

$$\lim_{\varepsilon\to 0}\left(a\left(t+\varepsilon\right) - a\left(t-\varepsilon\right)\right) = 0 \tag{19}$$

so that a must be continuous. [The neutral particle case corresponds to $q=0$ which means $\tau = 0$ from 13. In that case, the term $\tau\left(a\left(t+\varepsilon\right) - a\left(t-\varepsilon\right)\right)$ disappears and the argument above fails.]

Example. Now for a specific example. Suppose that the external force F is switched on at $t=0$, remains constant until $t=T$ and is then switched off. To determine the acceleration we need to solve 16 subject to the boundary condition that a is continuous everywhere. We have

$$a\left(t\right) = \begin{cases} a_0 e^{t/\tau} & t<0 \\ a_1 e^{t/\tau} + \frac{F}{m} & 0<t<T \\ a_2 e^{t/\tau} & t>T \end{cases} \tag{20}$$

Now we're faced with a problem. It would seem that since $F=0$ for both $t<0$ and $t>T$, there should be no acceleration in either of these zones. In particular, there clearly shouldn't be any acceleration for $t<0$

since the force hasn't been switched on yet, and there's no way the charge could know that a force is about to be switched on. So let's take $a_0 = 0$.

From continuity at $t = 0$ we therefore must have

$$a_1 + \frac{F}{m} = 0 \tag{21}$$

$$a_1 = -\frac{F}{m} \tag{22}$$

At $t = T$, we therefore must have

$$a_2 e^{T/\tau} = \frac{F}{m}\left(1 - e^{T/\tau}\right) \tag{23}$$

$$a_2 = \frac{F}{m}\left(e^{-T/\tau} - 1\right) \tag{24}$$

Having $a_2 \neq 0$ means that we get a spontaneously increasing (well, actually, increasing in the negative direction since $e^{-T/\tau} - 1 < 0$) acceleration for $t > T$, even though there is no external force at this time.

If we try to eliminate this runaway acceleration by starting with $a_2 = 0$ and working backwards, we get

$$a_1 e^{T/\tau} + \frac{F}{m} = 0 \tag{25}$$

$$a_1 = -e^{-T/\tau}\frac{F}{m} \tag{26}$$

$$a_0 = \frac{F}{m}\left(1 - e^{-T/\tau}\right) \tag{27}$$

Because $a_0 \neq 0$, the charge starts to accelerate *before* any force has been applied (actually, it accelerates starting at $t = -\infty$ which is clearly absurd). Not only that, but the rate of acceleration depends on the length of time T for which the force will be applied and the magnitude F of the force. This *preacceleration* violates causality, so something is clearly wrong.

In the case where we eliminate the runaway solution (when $a_2 = 0$), we can work out the acceleration and velocity (by integrating the acceleration and requiring continuity at all times):

$$a(t) = \begin{cases} \frac{F}{m}\left(1 - e^{-T/\tau}\right) e^{t/\tau} & t < 0 \\ \frac{F}{m}\left(1 - e^{(t-T)/\tau}\right) & 0 < t < T \\ 0 & t > T \end{cases} \tag{28}$$

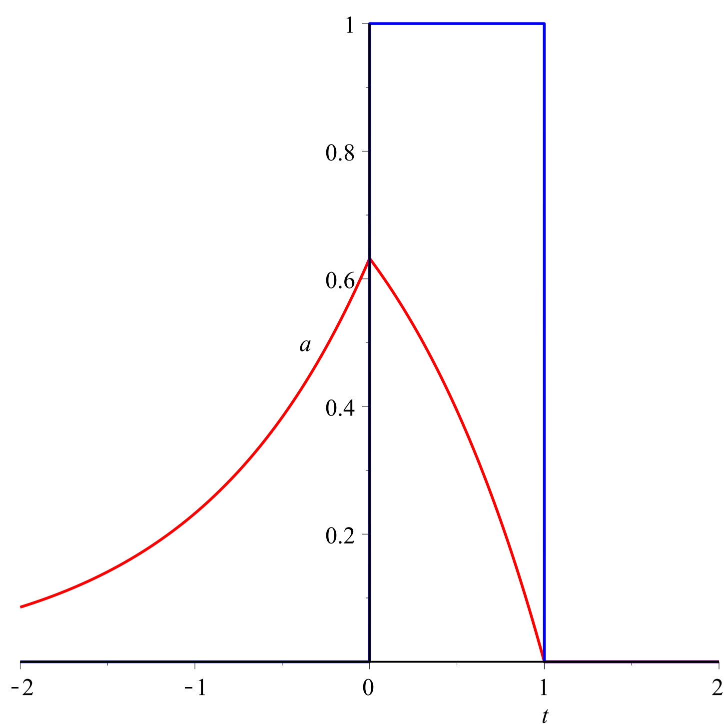
(29)
$$v(t)=\begin{cases}\frac{F\tau}{m}\left(1-e^{-T/\tau}\right)e^{t/\tau} & t<0\\ \frac{F}{m}\left(t-\tau e^{(t-T)/\tau}\right)+\frac{F\tau}{m} & 0<t<T\\ \frac{FT}{m} & t>T\end{cases}$$

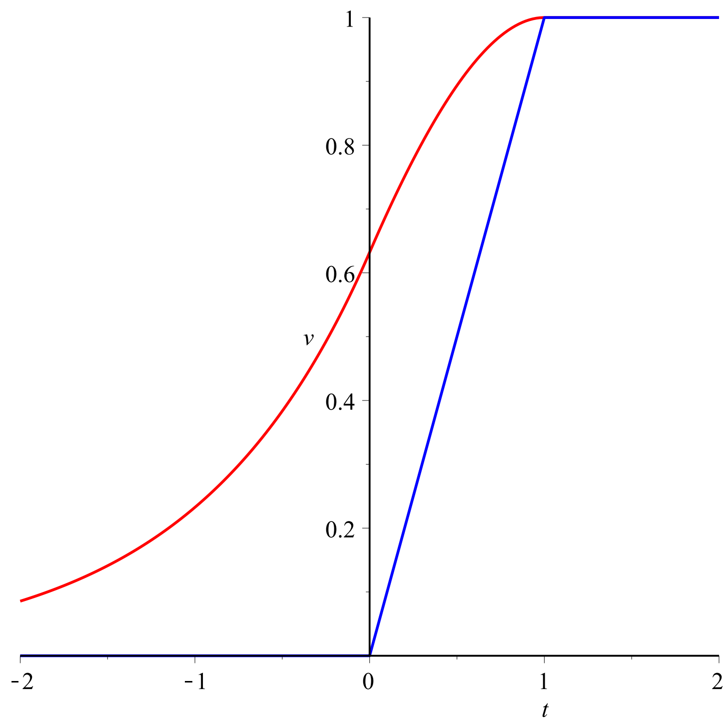
For an uncharged particle, the corresponding values are

(30)
$$a(t)=\begin{cases}0 & t<0\\ \frac{F}{m} & 0<t<T\\ 0 & t>T\end{cases}$$

(31)
$$v(t)=\begin{cases}0 & t<0\\ \frac{F}{m}t & 0<t<T\\ \frac{FT}{m} & t>T\end{cases}$$

Plotting these results (for $F=m=\tau=T=1$) we get (red curves for the charged particle; blue curves for the uncharged particle):





These dual problems of preacceleration and runaway acceleration have plagued the subject ever since the formula was first proposed. Clearly there is something wrong somewhere, but so far nobody has come up with a satisfactory solution. There is an alternative formula originally devised by Landau and Lifshitz as an approximation to the Abraham-Lorentz formula which states simply that

$$\mathbf{F}_{rad} = \tau \dot{\mathbf{F}}_{ext} \tag{32}$$

That is, the reaction force depends on the rate of change of the *external* force, so that if there is no external force, there is also no reaction force. This formula suffers from neither the preacceleration nor the runaway problems, but it raises the question of how an approximation can be more accurate than the formula it is approximating. Griffiths, Proctor and Schoeter consider the problem in more detail in a paper published after Griffiths's textbook (David J. Griffiths, Thomas C. Proctor and Darrell F. Schroeter, *American Journal of Physics*, **78**, 391 (2010)).

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