

RADIATION FROM A CHARGE ON A SPRING

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 21.

Here's another example of radiation from an accelerated charge. We hang a spring with spring constant k from the ceiling and attach a particle of mass m and charge q to its lower end, which has an equilibrium distance of h above the floor. If the particle is pulled down a distance d below the equilibrium and released at $t = 0$, we wish to find the intensity (average power per unit area) that hits the floor.

To solve this problem, we can use the Poynting vector for an arbitrary charge distribution that we found earlier:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \cong \frac{\mu_0 \dot{p}^2}{16\pi^2 c} \frac{\sin^2 \theta}{r^2} \hat{\mathbf{r}} \quad (1)$$

If we take the axis of the spring as the z axis and the equilibrium position as $z = 0$ and z increasing downwards, the dipole moment of the charge is

$$p(t) = qd \cos \omega t \quad (2)$$

$$\ddot{p}(t) = -qd\omega^2 \cos \omega t \quad (3)$$

In this coordinate system, the angle θ is the angle between the z axis and a point on the floor on a circle of radius R centred on the z axis. If the charge is at $z = 0$ then

$$\sin \theta = \frac{R}{\sqrt{R^2 + h^2}} \quad (4)$$

$$\cos \theta = \frac{h}{\sqrt{R^2 + h^2}} \quad (5)$$

$$r^2 = R^2 + h^2 \quad (6)$$

Therefore the Poynting vector is

$$\mathbf{S} = \frac{\mu_0 q^2 d^2 \omega^4 R^2 \cos^2 \omega t}{16\pi^2 c (R^2 + h^2)^2} \hat{\mathbf{r}} \quad (7)$$

This gives the power per unit area and unit time radiated along the r direction, pointing radially away from the charge. If we want the power received per unit area on the floor, we note that the angle between the normal to $\hat{\mathbf{r}}$ and the floor is equal to the angle between $\hat{\mathbf{r}}$ and the normal to the floor, the latter of which is the z axis. Thus the angle between the normal to $\hat{\mathbf{r}}$ and the floor is just θ , so that a unit area on the plane normal to $\hat{\mathbf{r}}$ spreads out to an area of $1/\cos\theta$ when projected onto the floor. Therefore the power per unit time per unit area of floor is

$$S_{floor} = \frac{S}{1/\cos\theta} \quad (8)$$

$$= \frac{\mu_0 q^2 d^2 \omega^4 R^2 \cos^2 \omega t}{16\pi^2 c (R^2 + h^2)^2} \cos\theta \quad (9)$$

$$= \frac{\mu_0 q^2 d^2 \omega^4 R^2 h \cos^2 \omega t}{16\pi^2 c (R^2 + h^2)^{5/2}} \quad (10)$$

Although we derived this for $z = 0$, it is a good approximation for all times provided that $d \ll h$, so the amplitude of oscillation is very small, since in that case the angle θ changes very little over a cycle.

The intensity of radiation on the floor is the time average of this over one cycle, and since the average of $\cos^2 \omega t$ over a cycle is $\frac{1}{2}$, we get

$$\langle S \rangle = \frac{\mu_0 q^2 d^2 \omega^4 R^2 h}{32\pi^2 c (R^2 + h^2)^{5/2}} \quad (11)$$

The radius R_{max} that receives the highest intensity of radiation is found by setting the derivative to zero:

$$\frac{d\langle S \rangle}{dR} = \frac{\mu_0 q^2 d^2 \omega^4 h}{32\pi^2 c} \left[\frac{2R}{(R^2 + h^2)^{5/2}} - \frac{5R^3}{(R^2 + h^2)^{7/2}} \right] \quad (12)$$

$$= 0 \quad (13)$$

$$R_{max} = \frac{\sqrt{6}h}{3} \quad (14)$$

If the floor is infinitely large, we can integrate 11 over all values of R to find the total power received by the floor:

$$P_{floor} = \frac{\mu_0 q^2 d^2 \omega^4 h}{32\pi^2 c} \int_0^\infty \frac{R^2}{(R^2 + h^2)^{5/2}} R dR \int_0^{2\pi} d\phi \quad (15)$$

The integral can be done by parts (integrate $\frac{R}{(R^2+h^2)^{5/2}}$ and differentiate R^2) so we get

$$\langle P_{floor} \rangle = \frac{\mu_0 q^2 d^2 \omega^4 h}{32\pi^2 c} \frac{4\pi}{3h} \quad (16)$$

$$= \frac{\mu_0 q^2 d^2 \omega^4}{24\pi c} \quad (17)$$

The total radiated power should be

$$P = \int \mathbf{S} \cdot d\mathbf{a} \cong \frac{\mu_0 \dot{p}^2}{6\pi c} \quad (18)$$

Averaged over a cycle this comes out to (using 3)

$$\langle P \rangle = \frac{\mu_0 \langle \dot{p}^2 \rangle}{6\pi c} \quad (19)$$

$$= \frac{\mu_0 \frac{1}{2} q^2 d^2 \omega^4}{6\pi c} \quad (20)$$

$$= \frac{\mu_0 q^2 d^2 \omega^4}{12\pi c} \quad (21)$$

Thus half the power gets absorbed by the floor, which is what we'd expect (the other half goes upwards and gets absorbed by the ceiling).

Finally, since the charge is radiating away energy, its total energy is decreasing so the amplitude of oscillation will decrease. The amount of power lost in time dt is $P dt$, which is equal to the negative change in energy, $-dE$. We therefore have, defining the amplitude as a function $A(t)$ of time:

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k z^2 \quad (22)$$

$$= \frac{1}{2} (m A^2 \omega^2 \sin^2 \omega t + k A^2 \cos^2 \omega t) \quad (23)$$

$$= \frac{1}{2} (m A^2 \omega^2 \sin^2 \omega t + m \omega^2 A^2 \cos^2 \omega t) \quad (24)$$

$$= \frac{1}{2} m A^2 \omega^2 \quad (25)$$

Therefore, the energy lost as the amplitude decreases is

$$-dE = -\frac{dE}{dA}dA \quad (26)$$

$$= -mA\omega^2 dA \quad (27)$$

The power for an amplitude A is

$$\langle P(A) \rangle = \frac{\mu_0 q^2 A^2 \omega^4}{12\pi c} \quad (28)$$

so

$$\frac{\mu_0 q^2 A^2 \omega^4}{12\pi c} dt = -mA\omega^2 dA \quad (29)$$

$$dt = -\frac{12\pi cm}{\mu_0 q^2 \omega^2} \frac{dA}{A} \quad (30)$$

To find how long it takes for the amplitude to drop from $A = d$ to $A = d/e$, we have

$$\int_0^T dt = -\frac{12\pi cm}{\mu_0 q^2 \omega^2} \int_d^{d/e} \frac{dA}{A} \quad (31)$$

$$T = \frac{12\pi cm}{\mu_0 q^2 \omega^2} \left[\ln d - \ln \frac{d}{e} \right] \quad (32)$$

$$T = \frac{12\pi cm}{\mu_0 q^2 \omega^2} \ln e \quad (33)$$

$$T = \frac{12\pi cm}{\mu_0 q^2 \omega^2} \quad (34)$$

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