FIELDS AND RADIATION FROM A TIME-DEPENDENT SHEET OF CURRENT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 24.

Suppose we have an infinite plane of current with surface current density $K(t)\hat{\mathbf{z}}$; that is, the current density at any instant of time is the same everywhere on the yz plane and flows in the z direction, but K can change with time.

To find the fields generated by this surface current, we need first to find the vector potential (we're assuming that the plane is electrically neutral so that the scalar potential $V=0$). From symmetry, $\mathbf{A}$ can depend only on x, the perpendicular distance from the yz plane. Suppose we're at some distance x on the x axis (it doesn't matter what y and z are, but for convenience we'll take $y=z=0$) from the plane. Then the potential is given by

$$\mathbf{A}(x,t)=\frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int\frac{K(t-\mathfrak{r}/c)}{\mathfrak{r}}d^2\mathbf{r} \tag{1}$$

where the integral is taken over the yz plane and for a point on the yz plane a distance r from the origin, we have

$$\mathfrak{r}=\sqrt{x^2+r^2} \tag{2}$$

It's easiest to do the integral using polar coordinates, since signals from all points at a given radius r on the yz plane that arrive at the observation point at time t left their points of origin at the same retarded time $t_r=t-\mathfrak{r}/c$. Therefore

$$\begin{aligned}\mathbf{A}(x,t) &= \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_0^\infty\int_0^{2\pi}\frac{K\left(t-\frac{\sqrt{x^2+r^2}}{c}\right)}{\sqrt{x^2+r^2}}r\,d\phi\,dr &(3)\\ &= \frac{\mu_0}{2}\hat{\mathbf{z}}\int_0^\infty\frac{K\left(t-\frac{\sqrt{x^2+r^2}}{c}\right)}{\sqrt{x^2+r^2}}r\,dr &(4)\end{aligned}$$

We can simplify the integral using the substitution (remember that x is a constant since it is the fixed point of observation):

$$\begin{aligned}
w &= \frac{\sqrt{x^2+r^2}}{c} && (5)\\
dw &= \frac{r}{c\sqrt{x^2+r^2}}dr && (6)\\
r=0 &\rightarrow w=\frac{x}{c} && (7)\\
r=\infty &\rightarrow w=\infty && (8)
\end{aligned}$$

So we get

$$\mathbf{A}(x,t)=\frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_{x/c}^{\infty}K(t-w)\,dw \tag{9}$$

Finally, to get rid of the inconvenient lower bound on the integral, we can use another substitution:

$$\begin{aligned}
u &= w-\frac{x}{c} && (10)\\
w=\frac{x}{c} &\rightarrow u=0 && (11)\\
w=\infty &\rightarrow u=\infty && (12)\\
\mathbf{A}(x,t) &= \frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^{\infty}K\left(t-\frac{x}{c}-u\right)du && (13)
\end{aligned}$$

From here, we can find the fields:

$$\begin{aligned}
\mathbf{E}(x,t) &= -\frac{\partial\mathbf{A}}{\partial t} && (14)\\
&= -\frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^{\infty}\dot{K}\left(t-\frac{x}{c}-u\right)du && (15)
\end{aligned}$$

However, from the chain rule

$$\begin{aligned}
\dot{K}\left(t-\frac{x}{c}-u\right) &= -\frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial u} && (16)\\
\mathbf{E}(x,t) &= \frac{\mu_0 c}{2}\hat{\mathbf{z}}\int_0^{\infty}\frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial u}du && (17)\\
&= \frac{\mu_0 c}{2}\hat{\mathbf{z}}\left[K(-\infty)-K\left(t-\frac{x}{c}\right)\right] && (18)
\end{aligned}$$

The original form 1 is actually valid only for finite current distributions, so we need to assume that at some finite past time t_0, $K(t)=0$ for all $t<t_0$. This means that the portion of the yz plane that contributes to $\mathbf{A}$ is always

finite, since if $t-\frac{\mathfrak{r}}{c}<t_0$, $K=0$, so $\mathfrak{r}$ must be restricted to $\mathfrak{r}<c(t-t_0)$. Therefore, the field is

$$\mathbf{E}(x,t)=-\frac{\mu_0 c}{2}K\left(t-\frac{x}{c}\right)\hat{\mathbf{z}} \tag{19}$$

The magnetic field is

$$\begin{aligned}\mathbf{B}(x,t) &= \nabla\times\mathbf{A} && (20)\\ &= -\frac{\partial A_z}{\partial x}\hat{\mathbf{y}} && (21)\\ &= -\frac{\mu_0 c}{2}\hat{\mathbf{y}}\int_0^\infty \frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial x}du && (22)\end{aligned}$$

From the chain rule

$$\frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial x}=\frac{1}{c}\frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial u} \tag{23}$$

so, again invoking the condition that $K(-\infty)=0$:

$$\begin{aligned}\mathbf{B}(x,t) &= -\frac{\mu_0}{2}\hat{\mathbf{y}}\int_0^\infty \frac{\partial K\left(t-\frac{x}{c}-u\right)}{\partial u}du && (24)\\ &= -\frac{\mu_0}{2}\hat{\mathbf{y}}\left[K(-\infty)-K\left(t-\frac{x}{c}\right)\right] && (25)\\ &= \frac{\mu_0}{2}K\left(t-\frac{x}{c}\right)\hat{\mathbf{y}} && (26)\end{aligned}$$

[To be precise, we should note that we've been assuming $x>0$, so that it represents the distance from the sheet of current rather than a strict x coordinate and thus the time $t_r=t-\frac{x}{c}$ is actually a retarded (earlier) time. If we look on the other side of the sheet where $x<0$ then we need to replace $K\left(t-\frac{x}{c}-u\right)$ by $K\left(t+\frac{x}{c}-u\right)$ to get the correct retarded time. This means that

$$\frac{\partial K\left(t+\frac{x}{c}-u\right)}{\partial x}=-\frac{1}{c}\frac{\partial K\left(t+\frac{x}{c}-u\right)}{\partial u} \tag{27}$$

which leads to

$$\mathbf{B}(x,t)=-\frac{\mu_0}{2}K\left(t+\frac{x}{c}\right)\hat{\mathbf{y}} \tag{28}$$

for $x<0$.]

Note that the electric field does *not* reduce to the correct value if we take K to be a constant for all time. In that case, since the sheet of current is electrically neutral and nothing is changing with time, $\mathbf{E}=0$. This is because we can't apply the analysis above to states where the current is not localized. [Curiously, though, we do get the correct answer from 18 if K is constant, since then $K(-\infty)-K\left(t-\frac{x}{c}\right)=0$. However, this logic doesn't work with $\mathbf{B}$, since for a constant surface current $\mathbf{B}=\frac{\mu_0 K}{2}\hat{\mathbf{y}}$ and taking K constant in 25 gives $\mathbf{B}=0$. The moral is, we just can't apply this analysis to infinite current distributions.]

The power radiated can be found from the Poynting vector:

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} &(29)\\ &= \frac{\mu_0 c}{4}K^2\left(t-\frac{x}{c}\right)\hat{\mathbf{x}} &(30)\end{aligned}$$

This is the energy per unit time per unit area radiated away from the sheet of current *on each side* of the sheet, so the total radiated energy is twice this. Also, the energy that leaves the surface of the plane (rather than the energy detected at some distance x from the plane) is found when $x=0$, so the total radiated energy is

$$E=\frac{\mu_0 c}{2}K^2(t) \tag{31}$$

Example 1. We can find the fields for a couple of special cases. First we consider

$$K(t)=\begin{cases}0 & t\le 0\\ K_0 & t>0\end{cases} \tag{32}$$

From 19 and 26 we have

$$\mathbf{E}(x,t)=\begin{cases}0 & t<\frac{x}{c}\\ -\frac{\mu_0 c}{2}K_0\hat{\mathbf{z}} & t>\frac{x}{c}\end{cases} \tag{33}$$

$$\mathbf{B}(x,t)=\begin{cases}0 & t<\frac{x}{c}\\ \frac{\mu_0}{2}K_0\hat{\mathbf{y}} & t>\frac{x}{c},\ x>0\\ -\frac{\mu_0}{2}K_0\hat{\mathbf{y}} & t>-\frac{x}{c},\ x<0\end{cases} \tag{34}$$

It may seem odd that $\mathbf{E}$ gets 'switched on' and remains constant for $t>\frac{x}{c}$, since once the current is on and constant, there is no electric field generated. However, the current gets switched on at one particular time *over the entire yz plane*, so due to retardation, the observer at point x receives news of this switching on from ever increasing circles on the yz plane as time progresses, so after the first news arrives (from the point directly below the observer, at a distance of x) there is a continual stream of news coming in that the current has switched on from circles that are ever further away, which results in a steady electric field.

It is also interesting that the magnetic field takes on the same value as that produced by an infinite sheet of constant current (see Griffiths, Example 5.8). This is because the magnetic field is determined solely by the point on the sheet directly below the observer. Contributions to $\mathbf{B}$ from other points on the sheet cancel each other out.

Example 2. This time, the current is linearly increasing

$$K(t)=\begin{cases}0 & t\leq 0\\ \alpha t & t>0\end{cases} \tag{35}$$

From 19 and 26 we have

$$\mathbf{E}(x,t)=\begin{cases}0 & t<\frac{x}{c}\\ -\frac{\mu_0 c\alpha}{2}\left(t-\frac{|x|}{c}\right)\hat{\mathbf{z}} & t>\frac{x}{c}\end{cases} \tag{36}$$

$$\mathbf{B}(x,t)=\begin{cases}0 & t<\frac{x}{c}\\ \frac{\mu_0\alpha}{2}\left(t-\frac{x}{c}\right)\hat{\mathbf{y}} & t>\frac{x}{c},\ x>0\\ -\frac{\mu_0\alpha}{2}\left(t+\frac{x}{c}\right)\hat{\mathbf{y}} & t>-\frac{x}{c},\ x<0\end{cases} \tag{37}$$