RADIATION REACTION: ENERGY CONSERVATION WITH A CONSTANT EXTERNAL FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 27.

Here, we'll return to the example of a constant force acting for a fixed time on a charge, and explore the energy distribution taking into account radiation reaction.

The external force F is switched on at $t=0$, remains constant until $t=T$ and is then switched off. The resulting acceleration and velocity are, if we take $a=0$ for $t>T$ and $v=0$ at $t=-\infty$:

$$a(t)=\begin{cases}\frac{F}{m}\left(1-e^{-T/\tau}\right)e^{t/\tau} & t<0\\ \frac{F}{m}\left(1-e^{(t-T)/\tau}\right) & 0<t<T\\ 0 & t>T\end{cases}\tag{1}$$

$$v(t)=\begin{cases}\frac{F\tau}{m}\left(1-e^{-T/\tau}\right)e^{t/\tau} & t<0\\ \frac{F}{m}\left(t-\tau e^{(t-T)/\tau}\right)+\frac{F\tau}{m} & 0<t<T\\ \frac{FT}{m} & t>T\end{cases}\tag{2}$$

First, we'll find how much work the external force does. Since the force acts only in the interval $0<t<T$, we have

$$\begin{aligned}W &= \int_0^T Fv\,dt &(3)\\ &= \int_0^T F\left[\frac{F}{m}\left(t-\tau e^{(t-T)/\tau}\right)+\frac{F\tau}{m}\right]dt &(4)\\ &= \frac{F^2}{m}\left[\frac{T^2}{2}-\tau^2\left(1-e^{-T/\tau}\right)\right]+\frac{F\tau T}{m} &(5)\end{aligned}$$

The charge radiates away energy whenever $a\ne0$, so it radiates for all $t<T$. The Larmor formula gives the rate at which energy is radiated:

$$P=\frac{\mu_0 q^2a^2}{6\pi c}=m\tau a^2\tag{6}$$

To get the total energy radiated R, we integrate P over time:

$$R = m\tau\int_{-\infty}^{0}\frac{F^2}{m^2}\left(1-e^{-T/\tau}\right)^2 e^{2t/\tau}dt + m\tau\int_0^T\frac{F^2}{m^2}\left(1-e^{(t-T)/\tau}\right)^2 dt \tag{7}$$

$$= \frac{F^2\tau}{m}\left[\frac{\tau}{2}\left(1-e^{-T/\tau}\right)^2 + \left(T-2\tau\left(1-e^{-T/\tau}\right)+\frac{\tau}{2}\left(1-e^{-2T/\tau}\right)\right)\right] \tag{8}$$

$$= \frac{F^2\tau}{m}\left[\frac{\tau}{2}+T-2\tau+\frac{\tau}{2}+e^{-T/\tau}\left(-\tau+2\tau\right)+e^{-2T/\tau}\left(\frac{\tau}{2}-\frac{\tau}{2}\right)\right] \tag{9}$$

$$= \frac{F^2\tau}{m}\left(T-\tau\left(1-e^{-T/\tau}\right)\right) \tag{10}$$

Since the final velocity is FT/m, the final kinetic energy is

$$K = \frac{F^2T^2}{2m} \tag{11}$$

By comparing terms, we see that

$$W = R + K \tag{12}$$

so that the work done by the force is divided between the energy radiated away and the final kinetic energy, so that energy is conserved.

Given that this problem suffers from preacceleration (violating causality) it's something of a miracle that energy is actually conserved. The charge seems to 'know' just how much to accelerate in anticipation of the force to be applied so that it can radiate away just the right amount of energy to balance out the work done and the final kinetic energy. There just *has* to be something wrong with the theory!

PINGBACKS

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