RADIATION REACTION WITH A DELTA-FUNCTION EXTERNAL FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 28.

Here's another example of applying an external force to a charge feeling the radiation reaction force. In general, a charge's acceleration obeys the differential equation

$$a = \tau \dot{a} + \frac{F}{m} \tag{1}$$

where F is the external force and

$$\tau \equiv \frac{\mu_0 q^2}{6\pi mc} \tag{2}$$

Suppose now that the force is a delta function:

$$F = k\delta(t) \tag{3}$$

for some constant k. In the earlier post, we showed that if F is finite everywhere, then a must be continuous everywhere. However, here F is not finite at $t = 0$. As before we start by integrating 1 over a small time interval around $t = 0$:

$$\int_{-\epsilon}^{\epsilon} a\, dt = \tau \left[a(\epsilon) - a(-\epsilon)\right] + \frac{1}{m}\int_{-\epsilon}^{\epsilon} F\, dt \tag{4}$$

Provided that a is finite everywhere, the integral on the LHS goes to zero as $\epsilon \to 0$ so we're left with

$$\begin{aligned}
\tau\Delta a &= -\frac{1}{m}\int_{-\epsilon}^{\epsilon}F\,dt && (5)\\
&= -\frac{k}{m}\int_{-\epsilon}^{\epsilon}\delta(t)\,dt && (6)\\
&= -\frac{k}{m} && (7)\\
\Delta a &= -\frac{k}{m\tau} && (8)
\end{aligned}$$

We can repeat the calculations we did earlier to check that energy is conserved here. Since $F=0$ everywhere except $t=0$, the general solution of 1 is

$$a(t)=\begin{cases}a_0e^{t/\tau} & t<0\\ a_1e^{t/\tau} & t>0\end{cases} \tag{9}$$

If we eliminate the runaway acceleration for $t>0$ by requiring $a_1=0$ then the condition 8 requires $a_0=k/m\tau$, so

$$a(t)=\begin{cases}\frac{k}{m\tau}e^{t/\tau} & t<0\\ 0 & t>0\end{cases} \tag{10}$$

By requiring $v=0$ at $t=-\infty$ and that v is continuous at $t=0$ we get

$$v(t)=\begin{cases}\frac{k}{m}e^{t/\tau} & t<0\\ \frac{k}{m} & t>0\end{cases} \tag{11}$$

The work done by the force is

$$\begin{aligned}
W &= \int_{-\infty}^{\infty}Fv\,dt && (12)\\
&= k\int_{-\infty}^{\infty}\delta(t)\,v\,dt && (13)\\
&= kv(0) && (14)\\
&= \frac{k^2}{m} && (15)
\end{aligned}$$

The energy radiated R is given by integrating the Larmor formula

$$P=\frac{\mu_0q^2a^2}{6\pi c}=m\tau a^2 \tag{16}$$

so we get

$$R = m\tau\int_{-\infty}^{0}\left(\frac{k}{m\tau}\right)^{2}e^{2t/\tau}dt \quad (17)$$

$$= \frac{k^{2}}{2m} \quad (18)$$

The final kinetic energy is

$$K = \frac{1}{2}m\frac{k^{2}}{m^{2}} = \frac{k^{2}}{2m} \quad (19)$$

Thus

$$W = R + K \quad (20)$$

and energy is conserved.

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