

TUNNELLING THROUGH A POTENTIAL BARRIER WITH THE RADIATION REACTION FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 29.

Besides giving rise to runaway acceleration and violation of causality, the radiation reaction force also predicts a classical version of tunnelling through a potential barrier, something you might think is confined to quantum mechanics. Suppose a charged particle travels in along the x axis, starting at $x = -\infty$ with some initial velocity v_i . Between $x = 0$ and $x = L$ there is a finite potential barrier of height U_0 .

In general, the charge's acceleration obeys the differential equation

$$(0.1) \quad a = \tau \dot{a} + \frac{F}{m}$$

where F is the external force and

$$(0.2) \quad \tau \equiv \frac{\mu_0 q^2}{6\pi m c}$$

Because the derivative of a step function is delta function and the force is the negative gradient of the potential, the force is

$$(0.3) \quad F = U_0 (-\delta(x) + \delta(x - L))$$

We've treated radiation reaction with a delta function force before, but in that case, the force was a delta function in *time* rather than space. However, we can envision the particle travelling in until it arrives at $x = 0$ at which point it feels a delta function force, then travelling along to $x = L$ where it feels another delta function force. Previously, to solve 0.1 around the time $t = 0$ we integrated the equation over a small interval about $t = 0$:

$$(0.4) \quad \int_{-\varepsilon}^{\varepsilon} a \, dt = \tau [a(\varepsilon) - a(-\varepsilon)] + \frac{1}{m} \int_{-\varepsilon}^{\varepsilon} F \, dt$$

If F were a delta function in time, the integral is easily done, as $\int_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1$. In our case, with the delta function a function of position, we can consider the position as a function of time and use the chain rule:

$$(0.5) \quad \int_{-\varepsilon}^{\varepsilon} \delta(x(t)) dt = \int_{-\varepsilon}^{\varepsilon} \delta(u) dt$$

where $u \equiv x(t)$ so that $du = \dot{x} dt = v dt$ and $dt = du/v$. Therefore

$$(0.6) \quad \int_{-\varepsilon}^{\varepsilon} \delta(x(t)) dt = \int_{-\varepsilon}^{\varepsilon} \frac{\delta(u)}{v} du$$

$$(0.7) \quad = \frac{1}{v(u=0)}$$

The velocity in the last line is evaluated at $u = 0$, which corresponds to $x = 0$. If we define our origin of time at this point, then this is also the velocity at $t = 0$, so we've converted the problem into the one we've already solved.

In our earlier solution, we saw that the acceleration for a force of $k\delta(t)$ has a discontinuity at $t = 0$, with $\Delta a = -k/m\tau$. Here, $k = -U_0/v_0$, where v_0 is the velocity at $t = 0$, so

$$(0.8) \quad \Delta a_0 = +\frac{U_0}{m\tau v_0}$$

Similarly, at $x = L$ we can say that the particle reaches this point at $t = T$ when the force is equal in magnitude but opposite in direction, so

$$(0.9) \quad \Delta a_T = -\frac{U_0}{m\tau v_T}$$

where v_T is the velocity at $t = T$. The general solution for the acceleration is therefore

$$(0.10) \quad a(t) = \begin{cases} a_0 e^{t/\tau} & t < 0 \\ a_1 e^{t/\tau} & 0 < t < T \\ a_2 e^{t/\tau} & t > T \end{cases}$$

If we set $a_2 = 0$ to prevent runaway acceleration for $t > T$, then we can apply 0.8 and 0.9 to determine a_1 and a_0 . We get at $t = T$

$$(0.11) \quad a_1 = \frac{U_0}{m\tau v_T} e^{-T/\tau}$$

Because $a_2 = 0$, there is no further acceleration after $t = T$, so the velocity at this time remains constant for all future times, and we can write it as v_f , the final velocity. Therefore

$$(0.12) \quad a_1 = \frac{U_0}{m\tau v_f} e^{-T/\tau}$$

At $t = 0$ we get

$$(0.13) \quad a_1 - a_0 = \frac{U_0}{m\tau v_0}$$

$$(0.14) \quad a_0 = \frac{U_0}{m\tau} \left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0} \right)$$

In summary,

$$(0.15) \quad a(t) = \begin{cases} \frac{U_0}{m\tau} \left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} & t < 0 \\ \frac{U_0}{m\tau v_f} e^{-T/\tau} e^{t/\tau} & 0 < t < T \\ 0 & t > T \end{cases}$$

Integrating this to get the velocity, we have

$$(0.16) \quad v(t) = \begin{cases} \frac{U_0}{m} \left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} + v_i & t < 0 \\ \frac{U_0}{mv_f} e^{-T/\tau} e^{t/\tau} + v_1 & 0 < t < T \\ v_f & t > T \end{cases}$$

Integrating again, we get the position

$$(0.17) \quad x(t) = \begin{cases} \frac{U_0\tau}{m} \left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} + v_i t + x_0 & t < 0 \\ \frac{U_0\tau}{mv_f} e^{-T/\tau} e^{t/\tau} + v_1 t + x_1 & 0 < t < T \\ v_f(t - T) + L & t > T \end{cases}$$

where in the last line, we've just imposed the condition that the particle moves at a constant velocity v_f starting from $x = L$ at $t = T$.

We now apply boundary conditions to find the various constants. First, to get x_0 we require $x(0) = 0$, which gives

$$(0.18) \quad x_0 = \frac{U_0\tau}{mv_0 v_f} (v_f - v_0 e^{-T/\tau})$$

Also using $t = 0$, we can find x_1 from the middle expression for $x(t)$:

$$(0.19) \quad x_1 = -\frac{U_0\tau}{mv_f}e^{-T/\tau}$$

Also from the middle expression for $x(t)$ we can find v_1 by requiring $x = L$ at $t = T$:

$$(0.20) \quad v_1 = \frac{1}{T} \left(L - \frac{U_0\tau}{mv_f} \left(1 - e^{-T/\tau} \right) \right)$$

We can now plug this into the middle expression for $v(t)$, set $t = 0$ and require the velocity to be v_0 to find v_0 :

$$(0.21) \quad v_0 = \frac{1}{T} \left(L + \frac{U_0}{mv_f} \left(-\tau + (\tau + T)e^{-T/\tau} \right) \right)$$

Substituting this back into the middle expression for $v(t)$ at $t = T$, when the velocity is v_f we find

$$(0.22) \quad L = v_f T - \frac{U_0}{mv_f} \left(T - \tau \left(1 - e^{-T/\tau} \right) \right)$$

Finally, we can find v_i by taking the first expression for $v(t)$, setting $t = 0$ and requiring the result to be equal to v_0 . This gives a rather unpleasant expression which can be simplified by collecting terms

(0.23)

$$v_i = \frac{m^2 v_f^4 - U_0 m \left(1 - e^{-T/\tau}\right) v_f^2 + U_0^2 \left(1 - e^{-T/\tau}\right)}{m v_f \left(m v_f^2 - U_0 \left(1 - e^{-T/\tau}\right)\right)}$$

(0.24)

$$= v_f - \frac{U_0}{m v_f} \left(\frac{U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right)$$

(0.25)

$$= v_f - \frac{U_0}{m v_f} \left(\frac{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right) - m v_f^2 - U_0 \left(e^{-T/\tau} - 1\right) + U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right)$$

(0.26)

$$= v_f - \frac{U_0}{m v_f} \left(1 - \frac{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right) - U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right)$$

(0.27)

$$= v_f - \frac{U_0}{m v_f} \left(1 - \frac{m v_f^2}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right)$$

(0.28)

$$= v_f - \frac{U_0}{m v_f} \left(1 - \frac{1}{1 + \frac{U_0}{m v_f^2} \left(e^{-T/\tau} - 1\right)} \right)$$

In the case where the final kinetic energy is half the barrier height, we have $m v_f^2 = U_0$ and we get

$$(0.29) \quad v_i = \frac{v_f}{e^{-T/\tau}}$$

This can be expressed in terms of the barrier length from 0.22:

$$(0.30) \quad L = v_f \tau \left(1 - e^{-T/\tau}\right)$$

$$(0.31) \quad v_i = \frac{v_f}{1 - L/v_f \tau}$$

We'd like to find a set of values such that both v_i and v_f are positive, and also the initial kinetic energy is less than U_0 . Choosing $L = v_f \tau/4$ gives

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$$(0.32) \quad v_i = \frac{4}{3}v_f$$

$$(0.33) \quad \frac{1}{2}mv_i^2 = \frac{8}{9}mv_f^2$$

$$(0.34) \quad = \frac{8}{9}U_0$$

Therefore the particle will actually tunnel through the barrier. However, given the crazy predictions arising from the reaction force, I'm not sure how much I would believe this result.