TUNNELLING THROUGH A POTENTIAL BARRIER WITH THE RADIATION REACTION FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 29.

Besides giving rise to runaway acceleration and violation of causality, the radiation reaction force also predicts a classical version of tunnelling through a potential barrier, something you might think is confined to quantum mechanics. Suppose a charged particle travels in along the x axis, starting at $x=-\infty$ with some initial velocity v_i. Between $x=0$ and $x=L$ there is a finite potential barrier of height U_0.

In general, the charge's acceleration obeys the differential equation

$$a=\tau\dot{a}+\frac{F}{m} \tag{1}$$

where F is the external force and

$$\tau\equiv\frac{\mu_0 q^2}{6\pi mc} \tag{2}$$

Because the derivative of a step function is delta function and the force is the negative gradient of the potential, the force is

$$F=U_0\left(-\delta\left(x\right)+\delta\left(x-L\right)\right) \tag{3}$$

We've treated radiation reaction with a delta function force before, but in that case, the force was a delta function in *time* rather than space. However, we can envision the particle travelling in until it arrives at $x=0$ at which point it feels a delta function force, then travelling along to $x=L$ where it feels another delta function force. Previously, to solve 1 around the time $t=0$ we integrated the equation over a small interval about $t=0$:

$$\int_{-\varepsilon}^{\varepsilon}a\,dt=\tau\left[a\left(\varepsilon\right)-a\left(-\varepsilon\right)\right]+\frac{1}{m}\int_{-\varepsilon}^{\varepsilon}F\,dt \tag{4}$$

If F were a delta function in time, the integral is easily done, as $\int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt = 1$. In our case, with the delta function a function of position, we can consider the position as a function of time and use the chain rule:

$$\int_{-\varepsilon}^{\varepsilon}\delta(x(t))\,dt = \int_{-\varepsilon}^{\varepsilon}\delta(u)\,dt \tag{5}$$

where $u \equiv x(t)$ so that $du = \dot{x}\,dt = v\,dt$ and $dt = du/v$. Therefore

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}\delta(x(t))\,dt &= \int_{-\varepsilon}^{\varepsilon}\frac{\delta(u)}{v}\,du && (6)\\ &= \frac{1}{v(u=0)} && (7)\end{aligned}$$

The velocity in the last line is evaluated at $u = 0$, which corresponds to $x = 0$. If we define our origin of time at this point, then this is also the velocity at $t = 0$, so we've converted the problem into the one we've already solved.

In our earlier solution, we saw that the acceleration for a force of $k\delta(t)$ has a discontinuity at $t = 0$, with $\Delta a = -k/m\tau$. Here, $k = -U_0/v_0$, where v_0 is the velocity at $t = 0$, so

$$\Delta a_0 = +\frac{U_0}{m\tau v_0} \tag{8}$$

Similarly, at $x = L$ we can say that the particle reaches this point at $t = T$ when the force is equal in magnitude but opposite in direction, so

$$\Delta a_T = -\frac{U_0}{m\tau v_T} \tag{9}$$

where v_T is the velocity at $t = T$. The general solution for the acceleration is therefore

$$a(t) = \begin{cases} a_0 e^{t/\tau} & t < 0 \\ a_1 e^{t/\tau} & 0 < t < T \\ a_2 e^{t/\tau} & t > T \end{cases} \tag{10}$$

If we set $a_2 = 0$ to prevent runaway acceleration for $t > T$, then we can apply 8 and 9 to determine a_1 and a_0. We get at $t = T$

$$a_1 = \frac{U_0}{m\tau v_T}e^{-T/\tau} \tag{11}$$

Because $a_2 = 0$, there is no further acceleration after $t = T$, so the velocity at this time remains constant for all future times, and we can write it as v_f, the final velocity. Therefore

$$a_1 = \frac{U_0}{m\tau v_f} e^{-T/\tau} \tag{12}$$

At $t = 0$ we get

$$\begin{aligned} a_1 - a_0 &= \frac{U_0}{m\tau v_0} && (13)\\ a_0 &= \frac{U_0}{m\tau}\left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0}\right) && (14) \end{aligned}$$

In summary,

$$a(t) = \begin{cases} \frac{U_0}{m\tau}\left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0}\right) e^{t/\tau} & t < 0 \\ \frac{U_0}{m\tau v_f} e^{-T/\tau} e^{t/\tau} & 0 < t < T \\ 0 & t > T \end{cases} \tag{15}$$

Integrating this to get the velocity, we have

$$v(t) = \begin{cases} \frac{U_0}{m}\left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0}\right) e^{t/\tau} + v_i & t < 0 \\ \frac{U_0}{m v_f} e^{-T/\tau} e^{t/\tau} + v_1 & 0 < t < T \\ v_f & t > T \end{cases} \tag{16}$$

Integrating again, we get the position

$$x(t) = \begin{cases} \frac{U_0\tau}{m}\left(\frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0}\right) e^{t/\tau} + v_i t + x_0 & t < 0 \\ \frac{U_0\tau}{m v_f} e^{-T/\tau} e^{t/\tau} + v_1 t + x_1 & 0 < t < T \\ v_f (t - T) + L & t > T \end{cases} \tag{17}$$

where in the last line, we've just imposed the condition that the particle moves at a constant velocity v_f starting from $x = L$ at $t = T$.

We now apply boundary conditions to find the various constants. First, to get x_0 we require $x(0) = 0$, which gives

$$x_0 = \frac{U_0\tau}{m v_0 v_f}\left(v_f - v_0 e^{-T/\tau}\right) \tag{18}$$

Also using $t = 0$, we can find x_1 from the middle expression for $x(t)$:

$$x_1 = -\frac{U_0\tau}{mv_f}e^{-T/\tau} \tag{19}$$

Also from the middle expression for $x(t)$ we can find v_1 by requiring $x = L$ at $t = T$:

$$v_1 = \frac{1}{T}\left(L - \frac{U_0\tau}{mv_f}\left(1 - e^{-T/\tau}\right)\right) \tag{20}$$

We can now plug this into the middle expression for $v(t)$, set $t = 0$ and require the velocity to be v_0 to find v_0:

$$v_0 = \frac{1}{T}\left(L + \frac{U_0}{mv_f}\left(-\tau + (\tau + T)\, e^{-T/\tau}\right)\right) \tag{21}$$

Substituting this back into the middle expression for $v(t)$ at $t = T$, when the velocity is v_f we find

$$L = v_f T - \frac{U_0}{mv_f}\left(T - \tau\left(1 - e^{-T/\tau}\right)\right) \tag{22}$$

Finally, we can find v_i by taking the first expression for $v(t)$, setting $t = 0$ and requiring the result to be equal to v_0. This gives a rather unpleasant expression which can be simplified by collecting terms

$$
\begin{aligned}
v_i &= \frac{m^2 v_f^4 - U_0 m \left(1 - e^{-T/\tau}\right) v_f^2 + U_0^2 \left(1 - e^{-T/\tau}\right)}{m v_f \left(m v_f^2 - U_0 \left(1 - e^{-T/\tau}\right)\right)} && (23) \\
&= v_f - \frac{U_0}{m v_f} \left(\frac{U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right) && (24) \\
&= v_f - \frac{U_0}{m v_f} \left(\frac{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right) - m v_f^2 - U_0 \left(e^{-T/\tau} - 1\right) + U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right) && (25) \\
&= v_f - \frac{U_0}{m v_f} \left(1 - \frac{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right) - U_0 \left(e^{-T/\tau} - 1\right)}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right) && (26) \\
&= v_f - \frac{U_0}{m v_f} \left(1 - \frac{m v_f^2}{m v_f^2 + U_0 \left(e^{-T/\tau} - 1\right)} \right) && (27) \\
&= v_f - \frac{U_0}{m v_f} \left(1 - \frac{1}{1 + \frac{U_0}{m v_f^2} \left(e^{-T/\tau} - 1\right)} \right) && (28)
\end{aligned}
$$

In the case where the final kinetic energy is half the barrier height, we have $mv_f^2 = U_0$ and we get

$$v_i = \frac{v_f}{e^{-T/\tau}} \tag{29}$$

This can be expressed in terms of the barrier length from 22:

$$
\begin{aligned}
L &= v_f \tau \left(1 - e^{-T/\tau}\right) && (30) \\
v_i &= \frac{v_f}{1 - L/v_f \tau} && (31)
\end{aligned}
$$

We'd like to find a set of values such that both v_i and v_f are positive, and also the initial kinetic energy is less than U_0. Choosing $L = v_f \tau/4$ gives

$$
\begin{aligned}
v_i &= \frac{4}{3}v_f && (32)\\
\frac{1}{2}mv_i^2 &= \frac{8}{9}mv_f^2 && (33)\\
&= \frac{8}{9}U_0 && (34)
\end{aligned}
$$

Therefore the particle will actually tunnel through the barrier. However, given the crazy predictions arising from the reaction force, I'm not sure how much I would believe this result.