PHYSICAL BASIS OF THE RADIATION REACTION FORCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 11, Post 30a.

We've seen that the radiation reaction force is given by the Abraham-Lorentz formula

$$\mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}} \tag{1}$$

Griffiths gives a derivation of this formula in his section 11.2.3 by considering a charge q split into a dumbbell of length d with its axis in the y direction moving along the x axis. A half-charge $q/2$ is at each end of the dumbbell, and the idea is that as the dumbbell moves along the x axis, each charge feels a force due to the fields emitted at the retarded time t_r by both the charge itself and the charge at the other end of the dumbbell. Griffiths's derivation, however, is for the special case where the dumbbell is momentarily at rest at the retarded time, which simplifies the calculations significantly. Here, we'll run through the derivation when the retarded velocity is not zero.

Due to the immense amount of algebra involved in this derivation, it makes sense to use Maple to do the calculations. However, even using Maple, the process is far from simple, so this post will be as much a tutorial on how to solve problems like this with Maple as it is on physics.

We start with the electric field due to a moving charge

$$\mathbf{E}(\mathbf{r},t) = \frac{q\mathfrak{r}}{4\pi\epsilon_0 (\mathfrak{r}\cdot\mathbf{u})^3}\left[\left(c^2-v^2\right)\mathbf{u}+\mathfrak{r}\times(\mathbf{u}\times\mathbf{a})\right] \tag{2}$$

where

$$\mathbf{u} = c\hat{\mathfrak{r}} - \mathbf{v} \tag{3}$$

and $\mathfrak{r}$ is the vector from the source charge *at the retarded time* to the destination charge *at the current time*.

If the dumbbell has moved a distance l along the x axis since the retarded time t_r, then if we're considering the field felt by a charge $q/2$ at one end of the dumbbell due to the charge at the other end emitted at the retarded time, we have

$$\begin{aligned} \mathfrak{r} &= \sqrt{l^2+d^2} && (4)\\ \boldsymbol{\mathfrak{r}} &= l\hat{\mathbf{x}}+d\hat{\mathbf{y}} && (5) \end{aligned}$$

Since all motion is along the x axis, we have

$$\begin{aligned} \boldsymbol{\mathfrak{r}}\cdot\mathbf{u} &= c\mathfrak{r}-lv && (6)\\ \boldsymbol{\mathfrak{r}}\cdot\mathbf{a} &= la && (7)\\ u_x &= \frac{cl}{\mathfrak{r}}-v && (8)\\ \boldsymbol{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a}) &= \mathbf{u}(\boldsymbol{\mathfrak{r}}\cdot\mathbf{a})-\mathbf{a}(\boldsymbol{\mathfrak{r}}\cdot\mathbf{u}) && (9)\\ &= la\mathbf{u}-(c\mathfrak{r}-lv)\,\mathbf{a} && (10) \end{aligned}$$

We'll refer to the top charge as charge 1 and the bottom charge as charge 2. From symmetry, the y component of $\mathbf{u}$ for the field due to charge 1 felt by charge 2 is equal and opposite to the same component for the field due to charge 2 felt by charge 1, so when added together, they cancel, and we need consider only the x component in what follows.

We can also ignore the magnetic forces since the magnetic field of a moving point charge is

$$\mathbf{B}(\mathbf{r},t)=\frac{1}{c}\hat{\boldsymbol{\mathfrak{r}}}\times\mathbf{E}(\mathbf{r},t) \tag{11}$$

Both $\mathbf{E}$ and $\boldsymbol{\mathfrak{r}}$ lie in the xy plane, so

$$\mathbf{B}=\left(\hat{\mathfrak{r}}_xE_y-\hat{\mathfrak{r}}_yE_x\right)\hat{\mathbf{z}} \tag{12}$$

For charge 2, both E_y and $\hat{\mathfrak{r}}_y$ are the negative of their counterparts for charge 1, while E_x and $\hat{\mathfrak{r}}_x$ are the same for both charges, so the magnetic field cancels out when added for the two charges.

Plugging in the above results and taking the x component we get for charge 1 (which is $q/2$) [I'll use r instead of $\mathfrak{r}$ in what follows since it's easier to type.]

$$E_x=\frac{q}{8\pi\epsilon_0c^3}\frac{r}{\left(r-\frac{lv}{c}\right)^3}\left[\left(\frac{cl}{r}-v\right)\left(\frac{c^2}{\gamma^2}+la\right)-ac\left(r-\frac{lv}{c}\right)\right] \tag{13}$$

where $\gamma\equiv 1/\sqrt{1-v^2/c^2}$.

What we want is an expansion of E_x as a power series in d, the distance between the charges (length of the dumbbell). In the process, we want to eliminate l and r from the expression. To do this, we first write things in terms of the time elapsed since the retarded time:

$$T \equiv t - t_r \tag{14}$$

First, we can expand the position

$$x(t) = x(t_r) + v(t_r)T + \frac{1}{2!}a(t_r)T^2 + \frac{1}{3!}\dot{a}(t_r)T^3 + \ldots \tag{15}$$

The distance travelled is $l = x(t) - x(t_r)$ so (dropping the dependence on t_r; until further notice, v, a and $\dot{a}$ are all assumed to be at the retarded time):

$$l = vT + \frac{a}{2}T^2 + \frac{\dot{a}}{6}T^3 \tag{16}$$

How do we know how many terms to keep in this expansion? Ultimately, we're interested in taking the limit as $d \to 0$, so we're looking for the constant term in the series expansion of 13. We know the final formula involves terms up to $\dot{a}$ so as a rule of thumb, we keep terms up to that point.

Things seem to be getting worse, in that we've introduced another parameter T, which we now need to get rid of. We note that T is the time taken for a signal to travel from charge 1 at the retarded time to charge 2 at the present time, so

$$\begin{aligned}
c^2T^2 &= r^2 = l^2 + d^2 && (17)\\
&= \left(vT + \frac{a}{2}T^2 + \frac{\dot{a}}{6}T^3\right)^2 + d^2 && (18)\\
&= d^2 + v^2T^2 + avT^3 + \left(\frac{v\dot{a}}{3} + \frac{a^2}{4}\right)T^4 && (19)\\
\frac{c^2}{\gamma^2}T^2 &= d^2 + avT^3 + \left(\frac{v\dot{a}}{3} + \frac{a^2}{4}\right)T^4 && (20)
\end{aligned}$$

Again, we've kept powers of T only up until the first term containing $\dot{a}$. Now we express T as a series in powers of d:

$$T = A_0 + A_1 d + A_2 d^2 + A_3 d^3 \tag{21}$$

By trial and error, we find that we need up to the cubic term to include a term with $\dot{a}$. We can now substitute this into 20 and equate powers of d on both sides. This is where it's useful to bring in Maple. We can use the command:

```
[code]
dist := c^2*T^2/gamma^2 = d^2+v*T^3*a+(1/3)*v*T^4*A+(1/4)*a^2*T^4;
Teq := subs(T = A__3*d^3+A__2*d^2+A__1*d+A__0, dist);
simplify(zip(`=`,
  [coeffs(select(t -> degree(t, d) <= 4, lhs(expand(Teq))), d)],
  [coeffs(select(t -> degree(t, d) <= 4, rhs(expand(Teq))), d)]));
As := solve(Tcoeffs, {A__0, A__1, A__2, A__3});
assign(As[1]);
[/code]
```

This Maple code defines an equation called 'dist' that is equivalent o 20, then substitutes 21 into it. The next command selects all the coefficients from terms with degree of d less than or equal to 4 from the LHS and RHS of the expanded equation and applies the $=$ operator between each pair. We then call **solve** to solve for the values of the A_i. The final **assign** assigns the A_is to the values found by solve. (The [1] means to take the first set of solutions; there are actually two sets of solutions but the second set contains negative values so we discard it.) The results are

$$\begin{aligned} A_0 &= 0 && (22)\\ A_1 &= \frac{\gamma}{c} && (23)\\ A_2 &= \frac{av\gamma^4}{c^4} && (24)\\ A_3 &= \frac{\gamma^5}{24c^7}\left(15a^2\gamma^2v^2+4\dot{a}vc^2+3a^2c^2\right) && (25)\\ &= \frac{\gamma^5}{24c^7}\left(4\dot{a}vc^2+3a^2\gamma^2\left(c^2+4v^2\right)\right) && (26) \end{aligned}$$

We can now plug these values back into 21 and then into 16 to get an expansion for l. Keeping only terms up to the first occurrence of $\dot{a}$ we use the Maple code

```
[code]
l := select(t -> degree(t, d) < 4, collect(expand(l), d));
l__2 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(l, d, 2)));
l__3 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(l, d, 3)));
l__3 := simplify(subs(c^2-v^2 = c^2/gamma^2, l__3));
[/code]
```

We also do a few tweaks and substitutions by hand to convert a few c^2-v^2 terms into γ terms until we get the result

$$l = \frac{\gamma v}{c} d + \frac{\gamma^4 a}{2c^2} d^2 + \frac{\gamma^5}{24c^5}\left(15a^2\gamma^2 v + 4\dot{a}c^2\right) d^3 \tag{27}$$

Since we've now got l as a series in d, we can use it to find expansions of the other factors in 13.

For r we get in Maple

```
[code]
r__s := series(r, d, 4);
r__1 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(r__s, d, 1)));
r__2 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(r__s, d, 2)));
r__2 := simplify(subs(c^2-v^2 = c^2/gamma^2, r__2));
r__3 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(r__s, d, 3)));
r__3 := simplify(subs(c^2-v^2 = c^2/gamma^2, r__3));
[/code]
```

With a few more **simplify** commands, we get

$$r = \gamma d + \frac{\gamma^4 va}{2c^3} d^2 + \frac{\gamma^5\left(3\,a^2c^2\gamma^2 + 12\,a^2\gamma^2 v^2 + 4\,\dot{a}c^2 v\right)}{24c^6} d^3 \tag{28}$$

The other terms in 13 can be worked out similarly by using Maple's **series** command together with a few **simplify**s:

$$\frac{cl}{r} - v = \frac{\gamma a}{2c} d + \frac{\gamma^2}{12c^4}\left(3a^2\gamma^2 v + 2\dot{a}c^2\right) d^2 \tag{29}$$

$$\frac{c^2}{\gamma^2} + la = \frac{c^2}{\gamma^2} + \frac{va\gamma\, d}{c} + \frac{\gamma^4 a^2 d^2}{2c^2} + \frac{\left(15\,a^3\gamma^2 v + 4\,a\dot{a}c^2\right)\gamma^5 d^3}{24c^5} \tag{30}$$

$$r - \frac{lv}{c} = \frac{\gamma\left(c^2 - v^2\right)}{c^2} d + \frac{\gamma^7 a^2\left(c^2 - v^2\right)}{8\,c^6} d^3 \tag{31}$$

$$\left(r - \frac{lv}{c}\right)^{-3} = \frac{\gamma^3}{d^3} - 3/8\,\frac{\gamma^9 a^2}{c^4 d} + \frac{3\,\gamma^{15} a^4 d}{32\,c^8} \tag{32}$$

Now we can put everything together to find E_x:

```
[code]
clrv := simplify(series(c*l/r-v, d, 5));
clrv__1 := coeff(clrv, d, 1);
clrv__1 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), clrv__1));
clrv__2 := simplify(subs(gamma = 1/sqrt(1-v^2/c^2), coeff(clrv, d, 2)));
clrv__2 := simplify((2*A*c^2/gamma^2+3*a^2*v)/(12*c^4/gamma^4));
clrv := clrv__2*d^2+clrv__1*d;
cgla := c^2/gamma^2+l*a;
rlvc := simplify(series(r-l*v/c, d, 4));
rlvc := simplify(subs(c^2-v^2 = c^2/gamma^2, rlvc));
```

```
rlvc := convert(rlvc, polynom);
n__1 := collect(expand(clrv*cgla), d);
n__2 := n__1-a*c*rlvc+n__1;
n__3 := select(t -> degree(t, d) < 4, collect(n__2, d));
d__1 := series(1/rlvc^3, d, 4);
d__1 := convert(d__1, polynom);
E := q*select(t -> degree(t, d) < 1, collect(r*n__3*d__1, d))/(8*Pi*epsil
[/code]
```

In the last line, we select only those terms of degree less than 1 in d, since we're interested only in terms that don't vanish as $d \to 0$. After a final **simplify** we get

$$E_x = -\frac{\gamma^3 aq}{16\pi\epsilon_0 c^2}\frac{1}{d} + \frac{q\gamma^4\left(3\,a^2\gamma^2 v + Ac^2\right)}{48c^5\pi\,\varepsilon_0} \tag{33}$$

The net force is $2\times(q/2)\,E_x$ since there is a force from both $q/2$ charges at the retarded time. Thus

$$F_x = -\frac{\gamma^3 aq^2}{16\pi\epsilon_0 c^2}\frac{1}{d} + \frac{q^2\gamma^4\left(3\,a^2\gamma^2 v + Ac^2\right)}{48c^5\pi\,\epsilon_0} \tag{34}$$

Somewhat embarrassingly, the first term blows up as $d \to 0$, but it is moved over to the LHS and incorporated as part of the particle's mass in a process called 'renormalization' (basically physics-speak for 'fudge'). This leaves the last term as the actual reaction force from one charge on the other.

To convert everything to the current time, we can use some more expansions (where a subscript t indicates current time):

$$v = v_t + \dot{v}_t\,(t_r - t) = v_t - a_t T \tag{35}$$

$$a = a_t - \dot{a}_t T \tag{36}$$

We take only up to first order terms in T, since the 'worst' power of d in F_x is d^{-1}, so a first order correction multiplied into a d^{-1} term will give an adjustment to the constant term, which is the term of interest. We need to transform not only the bare a and v terms in the force, but also the γ factor, since it depends on v. In Maple, we get (I use A to represent $\dot{a}$ in the Maple code):

```
[code]
E__0 := simplify(coeff(E, d, 0));
E__0t := subs({a = -A__t*T+a__t, v = -T*a__t+v__t}, subs(gamma = 1/sqrt(1
E__0ts := series(E__0t, d, 1);
simplify(E__0ts);
E__m1 := simplify(coeff(E, d, -1));
```

```
E__m1t := subs({a = -A__t*T+a__t, v = -T*a__t+v__t}, subs(gamma = 1/sqrt(
E__m1ts := series(E__m1t, d, 2);
E__m1ts := simplify(subs(c^2-v__t^2 = c^2/gamma^2, E__m1ts));
E__t := simplify(coeff(E__0ts, d, 0)+coeff(E__m1ts, d, 1));
E__t := simplify(subs(c^2-v__t^2 = c^2/gamma^2, E__t));
[/code]
```

We isolate the constant (d^0) coefficient from 33, substitute for γ and transform a and v and then do the same for the d^{-1} coefficient. The final result for the field is the sum of the order d term from the coefficient of d^{-1} (since the ds cancel out to give a constant term) plus the order d^0 term from the coefficient of d^0. The final result is

$$E_t = \frac{q\gamma^4\left(c^2\dot{a}_t + 12\,{a_t}^2\gamma^2 v_t + 3\,\dot{a}_t\,c^2\right)}{48c^5\pi\,\epsilon_0} \tag{37}$$

$$F_t = \frac{q^2\gamma^4\left(c^2\dot{a}_t + 12\,{a_t}^2\gamma^2 v_t + 3\,\dot{a}_t\,c^2\right)}{48c^5\pi\,\epsilon_0} \tag{38}$$

$$= \frac{q^2\gamma^4}{12\pi\epsilon_0 c^3}\left(\dot{a}_t + \frac{3a_t^2\gamma^2 v_t}{c^2}\right) \tag{39}$$

$$= \frac{\mu_0 q^2\gamma^4}{12\pi c}\left(\dot{a}_t + \frac{3a_t^2\gamma^2 v_t}{c^2}\right) \tag{40}$$

When the force of each end on itself is included, this doubles the answer, so we get

$$F = \frac{\mu_0 q^2\gamma^4}{6\pi c}\left(\dot{a}_t + \frac{3a_t^2\gamma^2 v_t}{c^2}\right) \tag{41}$$

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