VELOCITY ADDITION IN SPECIAL RELATIVITY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 12.3.

The velocity addition formula in special relativity is (if $\mathbf{v}_a$ and $\mathbf{v}_b$ are parallel):

$$v_r = \frac{v_a + v_b}{1 + v_a v_b / c^2} \tag{1}$$

Example 1. To get a feel for how small the correction to the classical formula $v_c = v_a + v_b$ is for everyday speeds, suppose $v_a = 5$ miles/hour $=$ 2.235 m s^{-1} and $v_b = 60$ miles/hour $= 26.82\text{ m s}^{-1}$. Since $v_a v_b / c^2$ in this case is very small, we can approximate v_r by

$$v_r \approx (v_a + v_b)\left(1 - \frac{v_a v_b}{c^2}\right) \tag{2}$$

The percentage error in the classical formula is then

$$\begin{aligned}
\Delta v &= \frac{v_c - v_r}{v_c} \times 100\% && (3)\\
&\approx \frac{(v_a + v_b)\left(v_a v_b / c^2\right)}{v_a + v_b} \times 100\% && (4)\\
&= \frac{v_a v_b}{c^2} \times 100\% && (5)\\
&= \frac{2.235 \times 26.82}{\left(3 \times 10^8\right)^2} \times 100\% && (6)\\
&= 6.66 \times 10^{-14}\% && (7)
\end{aligned}$$

It's not surprising that no relativistic effects are seen in the everyday world.

Example 2. Suppose you could run at $v_a = 0.5c$ (relative to the train) down the corridor of a train travelling at $v_b = 0.75c$. An observer on the ground would see your speed relative to the ground as

$$\begin{aligned} v &= \frac{0.5+0.75}{1+(0.5)(0.75)}c & (8) \\ &= 0.91c & (9) \end{aligned}$$

Even though the classical sum of velocities is greater than c, the relativistic formula still gives a result that is less than c.

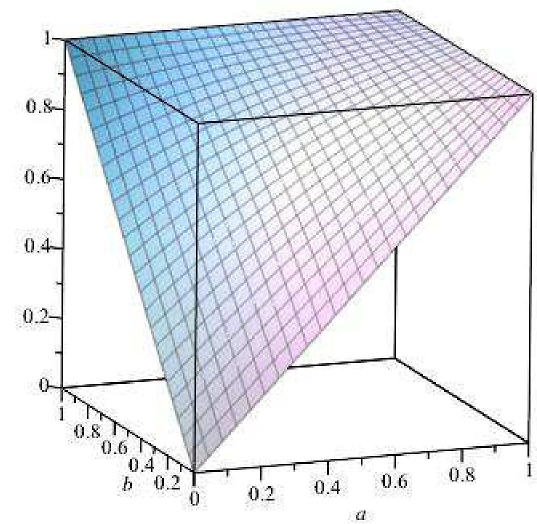
Example 3. The formula 1 always gives a result that is less than c. To prove this, we can simplify the notation by using velocities that are fractions of c so that $a \equiv v_a/c$ and $b \equiv v_b/c$ with the sum given by $s \equiv v_r/c$. Then

$$s = \frac{a+b}{1+ab} \tag{10}$$

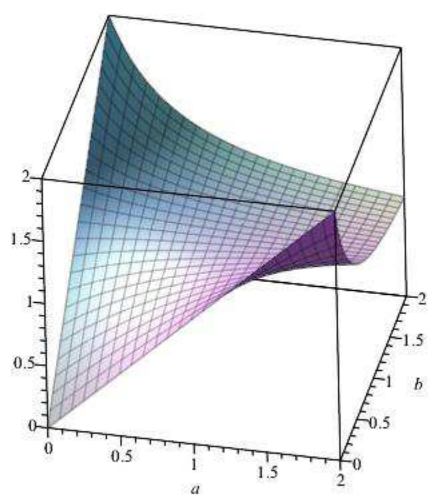
To check if there are any extrema in the region $0 \le a \le 1$, $0 \le b \le 1$ we can take the two partial derivatives and set them to zero:

$$\begin{aligned} \frac{\partial s}{\partial a} &= \frac{1}{1+ab} - \frac{(a+b)\,b}{(1+ab)^2} = 0 & (11) \\ \frac{\partial s}{\partial b} &= \frac{1}{1+ab} - \frac{(a+b)\,a}{(1+ab)^2} = 0 & (12) \end{aligned}$$

The only solution within the region is $a = b = 1$. We can see that this must be a maximum within the region, since along the border $a = 1$ or the border $b = 1$ we have $s = 1$, along the border $a = 0$ we have $s = b$ and along the border $b = 0$ we have $s = a$ so that $s \le 1$ on all borders. Thus $s \le 1$ everywhere in the region $0 \le a \le 1$, $0 \le b \le 1$. The surface 10 within this region looks like this:



The point $a=b=1$ is, however, actually a saddle point, as an expanded plot of the region $0\le a\le 2$, $0\le b\le 2$ reveals:



PINGBACKS

Pingback: Velocity addition formulas for all 3 directions

Pingback: Spectroscopic, eclipsing binary stars: mass, radius and temperature