LORENTZ (LENGTH) CONTRACTION: A SIMPLE EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 9.

In addition to time dilation, the other main kinematic prediction of special relativity is *length contraction*. The formula can be derived using a similar "light bulb in a train" experiment that we used to derive time dilation. As usual, we have a train moving at speed v along a straight track, and an observer T on the train and a second observer G on the ground. We have a light bulb at one end of the train's car and a mirror at the other end. If the length of the car, as measured by T, is Δx_T, then the time taken for the light to make a round trip from the bulb to the mirror and back is

$$\Delta t_T = 2\frac{\Delta x_T}{c} \tag{0.1}$$

To G, however, the the outward and return trips of the light take different times, due to the motion of the train. The outward journey takes

$$\Delta t_{Go} = \frac{\Delta x_G + v\Delta t_{Go}}{c} \tag{0.2}$$

and the return journey takes

$$\Delta t_{Gr} = \frac{\Delta x_G - v\Delta t_{Gr}}{c} \tag{0.3}$$

Note that we are assuming that the length of the car may be different to the two observers. We can solve these two equations for the times and get

$$\Delta t_{Go} = \frac{\Delta x_G}{c-v} \tag{0.4}$$

$$\Delta t_{Gr} = \frac{\Delta x_G}{c+v} \tag{0.5}$$

The total time measured by G is therefore

$$\Delta t_G = \Delta t_{Go} + \Delta t_{Gr} \tag{0.6}$$
$$= \Delta x_G \left(\frac{1}{c-v} + \frac{1}{c+v} \right) \tag{0.7}$$
$$= \Delta x_G \left(\frac{2c}{c^2 - v^2} \right) \tag{0.8}$$
$$= \frac{2\Delta x_G}{c} \gamma^2 \tag{0.9}$$

The two times are related by the time dilation formula. To apply this correctly, we need to note that T uses only the one clock to measure both the departure and arrival of the light, since these two events happen at the same place in his frame. Observer G must use two clocks, since the train moves relative to G between the events. Thus T's time is the proper time and must be less than G's time, so

$$\Delta t_G = \gamma \Delta t_T \tag{0.10}$$

Combining this with 0.1 and 0.9 we get

$$\frac{2\Delta x_G}{c} \gamma^2 = 2\gamma \frac{\Delta x_T}{c} \tag{0.11}$$
$$\Delta x_G = \frac{\Delta x_T}{\gamma} \tag{0.12}$$

Therefore the length of the car as measured by G is shorter than the length measured by T, by the factor γ. This is the length (or Lorentz) contraction effect.

Example. Suppose we have two spaceships A and B, and the rest length L_A of A is twice the rest length L_B of B. If B is moving at $v_B = c/2$ and A is moving at a speed v_A that makes A appear the same length as B to an observer at rest, how fast is A moving?

We must have

$$\frac{L_A}{\gamma_A} = \frac{L_B}{\gamma_B} \tag{0.13}$$

$$\frac{2}{\gamma_A} = \frac{1}{\gamma_B} \tag{0.14}$$

$$\frac{1}{\gamma_B^2} = 1-\frac{c^2/4}{c^2}=\frac{3}{4} \tag{0.15}$$

$$4\left(1-\frac{v_A^2}{c^2}\right) = \frac{3}{4} \tag{0.16}$$

$$v_A = \frac{\sqrt{13}}{4}c \tag{0.17}$$

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