LORENTZ CONTRACTION IN A ROTATING DISK: EHRENFEST'S AND BELL'S SPACESHIP PARADOXES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 11.

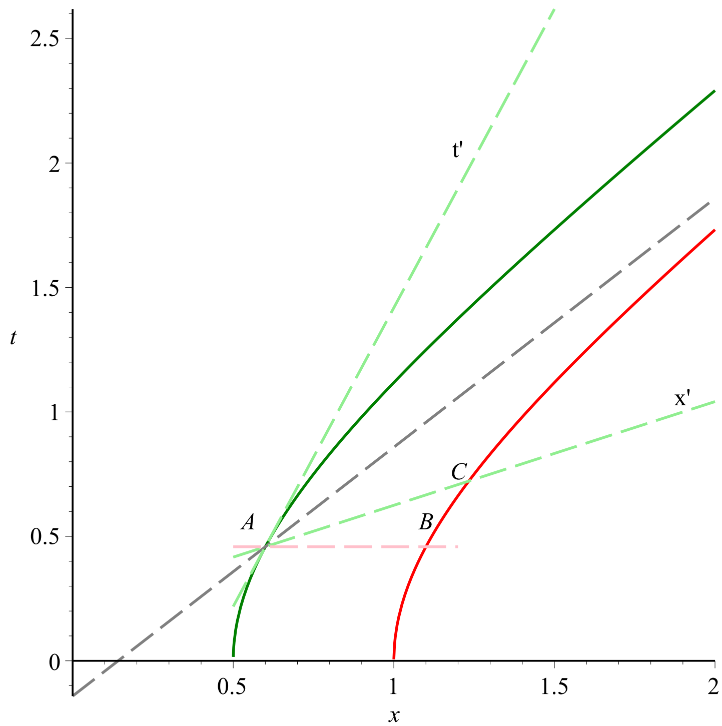
Kassner, Klaus, "Spatial geometry of the rotating disk and its non-rotating counterpart", *Am. J. Phys.*, **80**, 772 (2012).

In special relativity, lengths are contracted in the direction of motion, but not in perpendicular directions.A straightforward application of these formulas can, however, lead you astray if you're not careful. A case in point is that of a rotating disk of radius R. Points on the rim of the disk are moving parallel to the rim, so we would think that the rim of the disk must be contracted as seen by an observer at the centre of the disk. The radius of the disk, however, is perpendicular to the direction of motion at each point, so is not contracted. Thus the circumference of the disk would seem to be reduced to $2\pi R/\gamma$ while the diameter remains at $2R$, giving a ratio of π/γ between circumference and radius. This violates basic geometry (and is known as *Ehrenfest's paradox*).

The first point to note is that a rotating disk is *not* an inertial frame, since all points on the disk are experiencing centripetal acceleration towards the centre.

What actually happens is that the material in the disk physically stretches as it accelerates to its final rotation speed in such a way as to balance the Lorentz contraction and retain a ratio of π from circumference to diameter. In other words, Lorentz contraction is a real, physical effect that actually deforms the material. That this is so is most easily seen by examining *Bell's spaceship paradox*.

Suppose we have two spaceships, A and B, that accelerate from rest in such a way that the distance between them is always constant, as seen by an earthbound observer E (see diagram):



In the diagram, the solid green and red paths are parallel (well, they are meant to be anyway; the diagram is a bit skewed). The pink dashed line connecting A and B represents the positions of the two spacecraft at one specific time as measured by E. [The grey dashed line is the world line of a light beam leaving A.]

Now let's look at things from the point of view of an observer aboard A. A's coordinate axes are shown as the pale green dashed lines, for a particular point in the flight. At this time, according to A, ship B will be at point C, since points A and C have the same time coordinate in A's system. Although we can't compare the lengths of the lines AB and AC directly from the diagram since the units along the x and x' axes have different lengths, what we *can* say is that since C is further along ship B's world line than the point B, it has accelerated more than A so the distance between the two ships must be larger than the distance AB. If the two ships were connected by a string of length AB, then as the two ships accelerate, the string will break. Observer A thinks this is because B is getting further away, but observer E says it happens because as the string speeds up, it gets Lorentz contracted so it becomes shorter than the distance AB and therefore breaks.

To apply this to the rotating disk, consider two points close to each other on the rim. As the disk spins up, these two points are experiencing the same tangential acceleration, so the situation is essentially the same as with the two accelerating spaceships. As a result, one point on the rim sees the other point moving further away from it as long as the disk keeps accelerating. That is, the disk actually stretches as it accelerates.

We haven't actually proved that the amount of stretch is the correct amount to balance the Lorentz contraction and keep the ratio of circumference to diameter equal to π, but hopefully the qualitative explanation helps a bit.