LORENTZ TRANSFORMATIONS: FORWARD AND BACKWARD

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 12.

We can combine the effects of time dilation and length contraction to derive the Lorentz transformations. As usual, we align the two inertial frames $\mathcal{S}$ and $\bar{\mathcal{S}}$ (which moves to the right at speed v relative to $\mathcal{S}$) so that the x and $\bar{x}$ axes coincide at their origins also coincide at $t=\bar{t}=0$. Consider a point on the $\bar{x}$ axis (that, obviously, has coordinate $\bar{x}$ in the $\bar{\mathcal{S}}$ system). Due to length contraction, the distance from the origin of $\bar{\mathcal{S}}$ to $\bar{x}$ as measured in $\mathcal{S}$ is (remember that the measurement of a distance in $\mathcal{S}$ means that both endpoints of the distance are measured at the same time t, and that these two measurement events in $\mathcal{S}$ will be seen as occurring at *different* times $\bar{t}_1$ and $\bar{t}_2$ in the $\bar{\mathcal{S}}$ system):

$$d=\frac{\bar{x}}{\gamma} \tag{1}$$

so, since $\bar{\mathcal{S}}$ is moving at speed v relative to $\mathcal{S}$, the location of point $\bar{x}$ as measured by $\mathcal{S}$ is

$$\begin{aligned} x &= d+vt && (2)\\ &= \frac{\bar{x}}{\gamma}+vt && (3)\\ \bar{x} &= \gamma(x-vt) && (4) \end{aligned}$$

Obviously, we can invert the procedure by looking at the point of view of $\bar{\mathcal{S}}$. In this case, everything is the same, except that $\mathcal{S}$ is moving to the *left* with speed $-v$, so

$$x=\gamma(\bar{x}+v\bar{t}) \tag{5}$$

We can use this equation to eliminate $\bar{x}$ from 4 to get

$$\begin{aligned}
\frac{x}{\gamma}-v\bar{t} &= \gamma(x-vt) && (6)\\
\bar{t} &= \frac{1}{v}\left[\frac{x}{\gamma}-\gamma(x-vt)\right] && (7)\\
&= \frac{\gamma}{v}\left[x\left(1-\frac{v^2}{c^2}-1\right)+vt\right] && (8)\\
&= \gamma\left(t-\frac{xv}{c^2}\right) && (9)
\end{aligned}$$

Since directions perpendicular to the motion don't change, the full set of Lorentz transformations is

$$\begin{aligned}
\bar{x} &= \gamma(x-vt) && (10)\\
\bar{y} &= y && (11)\\
\bar{z} &= z && (12)\\
\bar{t} &= \gamma\left(t-\frac{xv}{c^2}\right) && (13)
\end{aligned}$$

As mentioned above, it's fairly obvious that the inverse transformations can be obtained by replacing v by $-v$:

$$\begin{aligned}
x &= \gamma(\bar{x}+v\bar{t}) && (14)\\
y &= \bar{y} && (15)\\
z &= \bar{z} && (16)\\
t &= \gamma\left(\bar{t}+\frac{\bar{x}v}{c^2}\right) && (17)
\end{aligned}$$

If you must, you can derive these from the original transformations. Starting with 4 and then 13:

$$x = \frac{\bar{x}}{\gamma}+vt \tag{18}$$

$$\bar{t} = \gamma\left(t-\frac{v\bar{x}}{\gamma c^2}-\frac{v^2}{c^2}t\right) \tag{19}$$

$$= \gamma\left(\frac{t}{\gamma^2}-\frac{v\bar{x}}{\gamma c^2}\right) \tag{20}$$

$$= \frac{t}{\gamma}-\frac{v\bar{x}}{c^2} \tag{21}$$

$$t = \gamma\left(\bar{t}+\frac{\bar{x}v}{c^2}\right) \tag{22}$$

Starting with 13 and then 4

$$t = \frac{\bar{t}}{\gamma}+\frac{vx}{c^2} \tag{23}$$

$$\bar{x} = \gamma\left(x-\frac{v\bar{t}}{\gamma}-\frac{v^2}{c^2}x\right) \tag{24}$$

$$= \gamma\left(\frac{x}{\gamma^2}-\frac{v\bar{t}}{\gamma}\right) \tag{25}$$

$$x = \gamma(\bar{x}+v\bar{t}) \tag{26}$$

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