

VELOCITY ADDITION FORMULAS FOR ALL 3 DIRECTIONS

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References: Griffiths, David J. (2007), *Introduction to Electrodynamics*, 3rd Edition; Pearson Education - Chapter 12, Problem 12.14.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.10.

The relativistic velocity addition formula is most easily derived from the Lorentz transformations:

$$(0.1) \quad \bar{x} = \gamma(x - vt)$$

$$(0.2) \quad \bar{t} = \gamma\left(t - \frac{xv}{c^2}\right)$$

As usual, system $\bar{\mathcal{S}}$ is moving to the right relative to system \mathcal{S} with velocity v . If an object is moving in the \mathcal{S} system with velocity $u = dx/dt$, then we can take differentials on both sides of the above equations to get

$$(0.3) \quad d\bar{x} = \gamma(dx - v dt)$$

$$(0.4) \quad d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$(0.5) \quad \bar{u} = \frac{d\bar{x}}{d\bar{t}}$$

$$(0.6) \quad = \frac{dx - v dt}{dt - \frac{v}{c^2}dx}$$

$$(0.7) \quad = \frac{(dx/dt) - v}{1 - \frac{v}{c^2}(dx/dt)}$$

$$(0.8) \quad = \frac{u - v}{1 - uv/c^2}$$

$$(0.9) \quad \dot{\bar{x}} = \frac{\dot{x} - v}{1 - \dot{x}v/c^2}$$

By symmetry (or algebra if you don't trust the physical argument) this relation can be inverted to give

$$(0.10) \quad u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2}$$

We can also derive the formulas for transforming velocity components perpendicular to the motion. Although perpendicular distances don't change under the Lorentz transformation, perpendicular *velocities* do, because they involve the ratio of perpendicular distance to time, and time *does* change from one system to another. We get, with a dot indicating a time derivative in the respective system:

$$(0.11) \quad \dot{y} = \frac{d\bar{y}}{d\bar{t}}$$

$$(0.12) \quad = \frac{dy}{\gamma \left(dt - \frac{v}{c^2} dx \right)}$$

$$(0.13) \quad = \frac{\dot{y}}{\gamma (1 - \dot{x}v/c^2)}$$

$$(0.14) \quad \dot{z} = \frac{\dot{z}}{\gamma (1 - \dot{x}v/c^2)}$$

Thus perpendicular velocities depend not only on the perpendicular velocity in the original system, but also on the horizontal velocity.

Example. Suppose we have a spotlight mounted on the roof of our usual train, and the spotlight is aimed to point towards the rear of the train, making an angle of θ with the roof of the train. If the train moves with speed v , what angle will a ground observer see the beam make with the train's roof?

In the train's system, the light's velocity components are

$$(0.15) \quad \dot{x} = -c \cos \theta$$

$$(0.16) \quad \dot{z} = c \sin \theta$$

In the ground system, we get from 0.10:

$$(0.17) \quad \dot{\bar{x}} = \frac{-c \cos \theta + v}{1 - v(\cos \theta)/c}$$

$$(0.18) \quad \dot{\bar{z}} = \frac{c \sin \theta}{\gamma(1 - v(\cos \theta)/c)}$$

$$(0.19) \quad \tan \bar{\theta} = -\frac{\dot{\bar{z}}}{\dot{\bar{x}}}$$

$$(0.20) \quad = \frac{c \sin \theta}{\gamma(c \cos \theta - v)}$$

As a check, we can work out $\dot{\bar{x}}^2 + \dot{\bar{z}}^2 = c^2$ (after simplifying). When the beam points directly up in the train's frame ($\theta = \frac{\pi}{2}$), it points slightly forward in the ground's frame. When $\cos \theta = \frac{v}{c}$, $\tan \bar{\theta} = \infty$ so the beam points directly up in the ground's frame.

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