VELOCITY ADDITION FORMULAS FOR ALL 3 DIRECTIONS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.14.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.10.

The relativistic velocity addition formula is most easily derived from the Lorentz transformations:

$$\bar{x} = \gamma(x - vt) \tag{1}$$

$$\bar{t} = \gamma\left(t - \frac{xv}{c^2}\right) \tag{2}$$

As usual, system $\bar{\mathscr{S}}$ is moving to the right relative to system $\mathscr{S}$ with velocity v. If an object is moving in the $\mathscr{S}$ system with velocity $u = dx/dt$, then we can take differentials on both sides of the above equations to get

$$d\bar{x} = \gamma(dx - v\,dt) \tag{3}$$

$$d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right) \tag{4}$$

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} \tag{5}$$

$$= \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} \tag{6}$$

$$= \frac{(dx/dt) - v}{1 - \frac{v}{c^2}(dx/dt)} \tag{7}$$

$$= \frac{u - v}{1 - uv/c^2} \tag{8}$$

$$\dot{\bar{x}} = \frac{\dot{x} - v}{1 - \dot{x}v/c^2} \tag{9}$$

By symmetry (or algebra if you don't trust the physical argument) this relation can be inverted to give

$$u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2} \tag{10}$$

We can also derive the formulas for transforming velocity components perpendicular to the motion. Although perpendicular distances don't change under the Lorentz transformation, perpendicular *velocities* do, because they involve the ratio of perpendicular distance to time, and time *does* change from one system to another. We get, with a dot indicating a time derivative in the respective system:

$$\dot{\bar{y}} = \frac{d\bar{y}}{d\bar{t}} \tag{11}$$

$$= \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} \tag{12}$$

$$= \frac{\dot{y}}{\gamma(1 - \dot{x}v/c^2)} \tag{13}$$

$$\dot{\bar{z}} = \frac{\dot{z}}{\gamma(1 - \dot{x}v/c^2)} \tag{14}$$

Thus perpendicular velocities depend not only on the perpendicular velocity in the original system, but also on the horizontal velocity.

Example. Suppose we have a spotlight mounted on the roof of our usual train, and the spotlight is aimed to point towards the rear of the train, making an angle of θ with the roof of the train. If the train moves with speed v, what angle will a ground observer see the beam make with the train's roof?

In the train's system, the light's velocity components are

$$\dot{x} = -c\cos\theta \tag{15}$$

$$\dot{z} = c\sin\theta \tag{16}$$

In the ground system, we get from 10:

$$\dot{\bar{x}} = \frac{-c\cos\theta+v}{1-v\left(\cos\theta\right)/c} \tag{17}$$

$$\dot{\bar{z}} = \frac{c\sin\theta}{\gamma\left(1-v\left(\cos\theta\right)/c\right)} \tag{18}$$

$$\tan\bar{\theta} = -\frac{\dot{\bar{z}}}{\dot{\bar{x}}} \tag{19}$$

$$= \frac{c\sin\theta}{\gamma\left(c\cos\theta-v\right)} \tag{20}$$

As a check, we can work out $\dot{\bar{x}}^2+\dot{\bar{z}}^2=c^2$ (after simplifying). When the beam points directly up in the train's frame ($\theta=\frac{\pi}{2}$), it points slightly forward in the ground's frame. When $\cos\theta=\frac{v}{c}$, $\tan\bar{\theta}=\infty$ so the beam points directly up in the ground's frame.

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