INVARIANCE OF SCALAR PRODUCT UNDER LORENTZ TRANSFORMATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.17.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.11.

Although the time and position of an event can change under Lorentz transformations, the scalar product of any two four-vectors is an invariant under a Lorentz transformation. That is

$$\bar{a}_i\bar{b}^i = a_ib^i \tag{0.1}$$

where for motion along the 1-axis (x axis) the transformations are

$$\begin{aligned}
\bar{a}^0 &= \gamma\left(a^0-\beta a^1\right) && (0.2)\\
\bar{a}^1 &= \gamma\left(a^1-\beta a^0\right) && (0.3)\\
\bar{a}^2 &= a^2 && (0.4)\\
\bar{a}^3 &= a^3 && (0.5)
\end{aligned}$$

and

$$\begin{aligned}
a_0 &= -a^0 && (0.6)\\
a_j &= a^j && (0.7)
\end{aligned}$$

for $j=1,2,3$. We can see this directly by calculation, using $\gamma^2 = 1/\left(1-\beta^2\right)$:

(0.8)
$$\bar{a}_i\bar{b}^i = -\gamma^2\left(a^0-\beta a^1\right)\left(b^0-\beta b^1\right)+\gamma^2\left(a^1-\beta a^0\right)\left(b^1-\beta b^0\right)+a^2b^2+a^3b^3$$
(0.9)
$$= \gamma^2\left[a^0b^0\left(-1+\beta^2\right)+a^1b^0\left(\beta-\beta\right)+a^0b^1\left(\beta-\beta\right)+a^1b^1\left(-\beta^2+1\right)\right]+a^2b^2+a^3b^3$$
(0.10)
$$= -a^0b^0+a^1b^1+a^2b^2+a^3b^3$$
(0.11)
$$= a_ib^i$$

In particular if a^i is the space-time four-vector of an event, or the difference between the four-vectors of two events, then

$$\bar{a}_i\bar{a}^i = a_ia^i \tag{0.12}$$

This leads to the *invariant interval* between two events:

$$\Delta s^2 \equiv -\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2 \tag{0.13}$$

PINGBACKS

Pingback: The invariant interval: some examples