

INVARIANCE OF SCALAR PRODUCT UNDER LORENTZ TRANSFORMATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.17.

Reference: Carroll, Bradley W. & Ostlie, Dale A. (2007), *An Introduction to Modern Astrophysics*, 2nd Edition; Pearson Education - Chapter 4, Problem 4.11.

Although the time and position of an event can change under Lorentz transformations, the scalar product of any two four-vectors is an invariant under a Lorentz transformation. That is

$$\bar{a}_i \bar{b}^i = a_i b^i \quad (1)$$

where for motion along the 1-axis (x axis) the transformations are

$$\bar{a}^0 = \gamma(a^0 - \beta a^1) \quad (2)$$

$$\bar{a}^1 = \gamma(a^1 - \beta a^0) \quad (3)$$

$$\bar{a}^2 = a^2 \quad (4)$$

$$\bar{a}^3 = a^3 \quad (5)$$

and

$$a_0 = -\bar{a}^0 \quad (6)$$

$$a_j = \bar{a}^j \quad (7)$$

for $j = 1, 2, 3$. We can see this directly by calculation, using $\gamma^2 = 1/(1 - \beta^2)$:

$$\bar{a}_i \bar{b}^i = -\gamma^2 (a^0 - \beta a^1) (b^0 - \beta b^1) + \gamma^2 (a^1 - \beta a^0) (b^1 - \beta b^0) + a^2 b^2 + a^3 b^3 \quad (8)$$

$$= \gamma^2 [a^0 b^0 (-1 + \beta^2) + a^1 b^0 (\beta - \beta) + a^0 b^1 (\beta - \beta) + a^1 b^1 (-\beta^2 + 1)] + a^2 b^2 + a^3 b^3 \quad (9)$$

$$= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \quad (10)$$

$$= a_i b^i \quad (11)$$

In particular if a^i is the space-time four-vector of an event, or the difference between the four-vectors of two events, then

$$\bar{a}_i \bar{a}^i = a_i a^i \quad (12)$$

This leads to the *invariant interval* between two events:

$$\Delta s^2 \equiv -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \quad (13)$$

PINGBACKS

Pingback: The invariant interval: some examples