

## COMPOUND LORENTZ TRANSFORMATIONS

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 18.

The Lorentz transformations can be written in matrix form as

$$\Lambda_x = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

where the 0 ( $ct$ ) component is the first row and first column, followed by the 1, 2, and 3 directions in order. This matrix is for relative motion along the 1 axis.

The Galilean transformations can be written as a matrix as well, where the first coordinate is just  $t$  rather than  $ct$ :

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

or if we want to use the same symbols as in the Lorentz case, where the top row of  $\Gamma$  is a  $ct$  coordinate, we can write

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

The Lorentz transformation along the 2 ( $y$ ) axis is obtained by putting the transformation terms in row and column 2:

$$\Lambda_y = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

If we apply a Lorentz transformation first in the  $x$  and then in the  $y$  direction (with different relative velocities), we get the compound matrix:

$$\Lambda_y \Lambda_x = \begin{bmatrix} \gamma_y & 0 & -\beta_y \gamma_y & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_x & -\beta_x \gamma_x & 0 & 0 \\ -\beta_x \gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} \gamma_y \gamma_x & -\gamma_y \gamma_x \beta_x & -\beta_y \gamma_y & 0 \\ -\beta_x \gamma_x & \gamma_x & 0 & 0 \\ -\beta_y \gamma_y \gamma_x & \gamma_y \gamma_x \beta_x \beta_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

Note that although  $\Lambda_x$  and  $\Lambda_y$  are both symmetric, their product is not. This means that applying the transformations in the opposite order gives a different result.

$$\Lambda_x \Lambda_y = \Lambda_x^T \Lambda_y^T \quad (7)$$

$$= (\Lambda_y \Lambda_x)^T \quad (8)$$

$$= \begin{bmatrix} \gamma_y \gamma_x & -\beta_x \gamma_x & -\beta_y \gamma_y \gamma_x & 0 \\ -\gamma_y \gamma_x \beta_x & \gamma_x & \gamma_y \gamma_x \beta_x \beta_y & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$