

THE INVARIANT INTERVAL: SOME EXAMPLES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 20-21.

The invariant interval in special relativity is the scalar product of the interval between two events with itself:

$$(1) \quad \Delta s^2 \equiv (\Delta x)_i (\Delta x)^i$$

Since Δx is the difference of two four-vectors, it too is a four-vector so the invariance under Lorentz transformations follows from that fact.

Because the 0 term is negative and the other three terms are positive, Δs^2 can be negative, zero or positive. This gives three possible types of pairs of events:

- (1) **Timelike:** If $\Delta s^2 < 0$, then it is possible to find a frame in which the two events occur at the same spatial point, but at different times, since it is the time component $-(\Delta x^0)^2$ which is negative.
- (2) **Lightlike:** If $\Delta s^2 = 0$ then $c^2 (\Delta t)^2 = \Delta x^2$ (if the motion is along the x axis; the argument is similar for arbitrary directions), so the events can be connected by a light signal.
- (3) **Spacelike:** If $\Delta s^2 > 0$, then it is possible to find a frame in which the two events occur at the same time but at different places. Different observers may disagree about which event occurs first.

Example 1. In system \mathcal{S} , an event A happens at $(ct, x, y, z) = (15, 5, 3, 0)$ and B happens at $(5, 10, 8, 0)$. The interval between them is

$$(2) \quad \Delta s^2 = -100 + 25 + 25 + 0 = -50 < 0$$

so the interval is timelike. There is no frame in which A and B occur simultaneously. However, there is a frame where they occur at the same point. To find this frame, it's easiest to orient the coordinates so that the x axis is along the line joining the events, which points in the direction $\hat{x} + \hat{y}$, and to redefine the origin so that $x = \bar{x} = 0$ and $t = \bar{t} = 0$ when A occurs. In \mathcal{S} , the events are separated by a distance of $5\sqrt{2}$, so in the new coordinate system (at rest relative to \mathcal{S}) we have

$$(3) \quad A = (0, 0, 0, 0)$$

$$(4) \quad B = (-10, 5\sqrt{2}, 0, 0)$$

We then need to find β such that in the frame $\bar{\mathcal{S}}$, $\bar{x}_B = 0$, so using a Lorentz transformation, we have

$$(5) \quad \bar{x}_B = 0 = \gamma(5\sqrt{2} + 10\beta)$$

$$(6) \quad \beta = -\frac{\sqrt{2}}{2}$$

Therefore the velocity of $\bar{\mathcal{S}}$ relative to our original frame \mathcal{S} is

$$(7) \quad \mathbf{v} = -\frac{c}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$$

Example 2. Now we take $A = (1, 2, 0, 0)$ and $B = (3, 5, 0, 0)$. The interval is

$$(8) \quad \Delta s^2 = -4 + 9 = 5 > 0$$

so the interval is spacelike. Since both events are already on the x axis, to find a frame in which the events occur at the same time, we change the origin to event A , giving

$$(9) \quad A = (0, 0, 0, 0)$$

$$(10) \quad B = (2, 3, 0, 0)$$

We now use a Lorentz transformation on the time to find β such that $\bar{t}_B = 0$:

$$(11) \quad \bar{t}_B = 0 = \gamma(t_B - \beta x_B)$$

$$(12) \quad \beta = \frac{t_B}{x_B}$$

$$(13) \quad = \frac{2}{3}$$

Example 3. The previous example is easily generalized to the case where $A = (t_A, x_A, 0, 0)$ and $B = (t_B, x_B, 0, 0)$. We redefine the origin to be at A , giving B coordinates of $B' = (t_B - t_A, x_B - x_A, 0, 0)$. Assuming the interval

is spacelike, the velocity of the frame in which A and B are simultaneous is found from

$$(14) \quad \bar{t}_B = 0 = \gamma(t_B - t_A - \beta(x_B - x_A))$$

$$(15) \quad \beta = \frac{t_B - t_A}{x_B - x_A}$$

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