THE INVARIANT INTERVAL: SOME EXAMPLES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 20-21.

The invariant interval in special relativity is the scalar product of the interval between two events with itself:

$$\Delta s^2 \equiv (\Delta x)_i (\Delta x)^i \tag{0.1}$$

Since Δx is the difference of two four-vectors, it too is a four-vector so the invariance under Lorentz transformations follows from that fact.

Because the 0 term is negative and the other three terms are positive, Δs^2 can be negative, zero or positive. This gives three possible types of pairs of events:

(1) **Timelike:** If $\Delta s^2 < 0$, then it is possible to find a frame in which the two events occur at the same spatial point, but at different times, since it is the time component $-\left(\Delta x^0\right)^2$ which is negative.

(2) **Lightlike:** If $\Delta s^2 = 0$ then $c^2 (\Delta t)^2 = \Delta x^2$ (if the motion is along the x axis; the argument is similar for arbitrary directions), so the events can be connected by a light signal.

(3) **Spacelike:** If $\Delta s^2 > 0$, then it is possible to find a frame in which the two events occur at the same time but at different places. Different observers may disagree about which event occurs first.

Example 1. In system $\mathscr{S}$, an event A happens at $(ct, x, y, z) = (15, 5, 3, 0)$ and B happens at $(5, 10, 8, 0)$. The interval between them is

$$\Delta s^2 = -100 + 25 + 25 + 0 = -50 < 0 \tag{0.2}$$

so the interval is timelike. There is no frame in which A and B occur simultaneously. However, there is a frame where they occur at the same point. To find this frame, it's easiest to orient the coordinates so that the x axis is along the line joining the events, which points in the direction $\hat{\mathbf{x}} + \hat{\mathbf{y}}$, and to redefine the origin so that $x = \bar{x} = 0$ and $t = \bar{t} = 0$ when A occurs. In $\mathscr{S}$, the events are separated by a distance of $5\sqrt{2}$, so in the new coordinate system (at rest relative to $\mathscr{S}$) we have

$$A = (0,0,0,0) \tag{0.3}$$
$$B = \left(-10, 5\sqrt{2}, 0, 0\right) \tag{0.4}$$

We then need to find β such that in the frame $\bar{\mathscr{S}}$, $\bar{x}_B = 0$, so using a Lorentz transformation, we have

$$\bar{x}_B = 0 = \gamma\left(5\sqrt{2} + 10\beta\right) \tag{0.5}$$
$$\beta = -\frac{\sqrt{2}}{2} \tag{0.6}$$

Therefore the velocity of $\bar{\mathscr{S}}$ relative to our original frame $\mathscr{S}$ is

$$\mathbf{v} = -\frac{c}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \tag{0.7}$$

Example 2. Now we take $A = (1,2,0,0)$ and $B = (3,5,0,0)$. The interval is

$$\Delta s^2 = -4 + 9 = 5 > 0 \tag{0.8}$$

so the interval is spacelike. Since both events are already on the x axis, to find a frame in which the events occur at the same time, we change the origin to event A, giving

$$A = (0,0,0,0) \tag{0.9}$$
$$B = (2,3,0,0) \tag{0.10}$$

We now use a Lorentz transformation on the time to find β such that $\bar{t}_B = 0$:

$$\bar{t}_B = 0 = \gamma(t_B - \beta x_B) \tag{0.11}$$
$$\beta = \frac{t_B}{x_B} \tag{0.12}$$
$$= \frac{2}{3} \tag{0.13}$$

Example 3. The previous example is easily generalized to the case where $A = (t_A, x_A, 0, 0)$ and $B = (t_B, x_B, 0, 0)$. We redefine the origin to be at A, giving B coordinates of $B' = (t_B - t_A, x_B - x_A, 0, 0)$. Assuming the interval is spacelike, the velocity of the frame in which A and B are simultaneous is found from

$$\bar{t}_B = 0 = \gamma\left(t_B - t_A - \beta\left(x_B - x_A\right)\right) \tag{0.14}$$

$$\beta = \frac{t_B - t_A}{x_B - x_A} \tag{0.15}$$

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