SPACETIME DIAGRAMS: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 23.

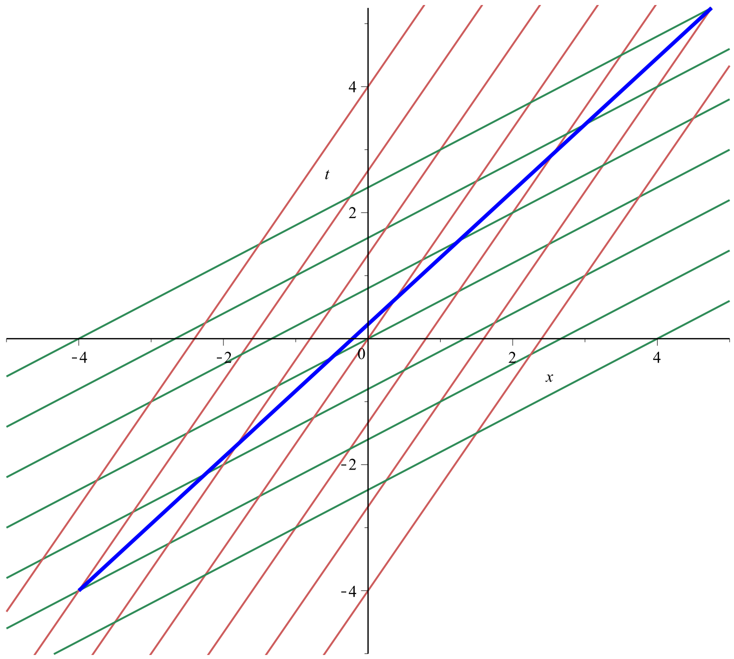
As an example of a spacetime diagram, suppose we have our usual two inertial frames with $\mathcal{S}$ at rest relative to observer A and $\bar{\mathcal{S}}$ moving at speed $\beta = \frac{3}{5}$ in the $+x$ direction, with the origins of the two systems coinciding as usual.

We'd like to plot the lines of constant $\bar{t}$ and $\bar{x}$ on a spacetime diagram. Using Lorentz transformations we have

$$ct = \frac{x}{\beta} - \frac{\bar{x}}{\beta\gamma} \tag{1}$$

$$ct = \beta x + \frac{c\bar{t}}{\gamma} \tag{2}$$

For various values of $\bar{x}$ and $\bar{t}$, these two equations give two sets of parallel lines. The first equation gives lines with slope $1/\beta$ and ct-intercepts of $-\bar{x}/\beta\gamma$, while the second equation gives lines with slope β and ct-intercepts of $c\bar{t}/\gamma$. We can plot a few lines from each set as shown:



The red lines are lines of constant $\bar{x}$ with the top line corresponding to $\bar{x} = -3$ and the bottom line to $\bar{x} = +3$, in steps of 1. The green lines are lines of constant $\bar{t}$ with the bottom line corresponding to $\bar{t} = -3$ and the top line to $\bar{t} = +3$, again in steps of 1.

The thick blue line represents the world line of an object that starts at $(\bar{t},\bar{x}) = (-2,-2)$ and moves to $(\bar{t},\bar{x}) = (3,2)$. We can find its velocity in $\mathcal{S}$ by taking its slope on the graph. Finding the exact values of x and t is difficult by eyeballing a graph, so we can 'cheat' a bit and use the Lorentz transformations to find the corresponding values. We get for the starting point:

$$
\begin{aligned}
ct_1 &= \gamma(c\bar{t} + \beta\bar{x}) && (3)\\
&= \frac{5}{4}\left(-2 - \frac{3}{5}2\right) && (4)\\
&= -4 && (5)\\
x_1 &= \gamma(\bar{x} + \beta c\bar{t}) && (6)\\
&= -4 && (7)
\end{aligned}
$$

and for the end point:

$$
\begin{aligned}
ct_2 &= \gamma(c\bar{t} + \beta\bar{x}) && (8)\\
&= \frac{5}{4}\left(3 + \frac{3}{5}2\right) && (9)\\
&= \frac{21}{4} && (10)\\
x_2 &= \gamma(\bar{x} + \beta c\bar{t}) && (11)\\
&= \frac{19}{4} && (12)
\end{aligned}
$$

The velocity is then

$$
\begin{aligned}
v &= \frac{\Delta x}{\Delta t} && (13)\\
&= \frac{35}{37}c && (14)
\end{aligned}
$$

We can check this using the velocity addition formula. Its velocity in $\bar{S}$ is

$$\bar{v} = \frac{\Delta\bar{x}}{\Delta\bar{t}} = \frac{4}{5}c \tag{15}$$

so

$$v = \frac{\frac{4}{5} + \frac{3}{5}}{1 + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)}c = \frac{35}{37}c \tag{16}$$

so it checks out.