

FOUR-VELOCITY AGAIN

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 24.

Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.

The four-velocity (or *proper velocity* as Griffiths calls it) is defined using the symbol η (which is used by Moore for the flat space metric) as the derivative of the four-position with respect to *proper* time:

$$(1) \quad \eta^i \equiv \frac{dx^i}{d\tau}$$

Ordinary velocity (which Griffiths calls u ; Moore uses u for the *four*-velocity. As I say, it gets confusing.) is defined as the derivative of the four-position with respect to the time component of the four-position:

$$(2) \quad u^i = \frac{dx^i}{dx^0} c$$

where the factor of c is there to cancel out the c in $x^0 = ct$. Since proper time and coordinate time are related by

$$(3) \quad d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt$$

we have

$$(4) \quad dx^0 = c dt$$

$$(5) \quad = \frac{c d\tau}{\sqrt{1 - u^2/c^2}}$$

where $u^2 = \sum_{i=1}^3 (u^i)^2$, that is, the square magnitude of the spatial ordinary velocity. Therefore

$$(6) \quad \eta^i = \frac{u^i}{\sqrt{1 - u^2/c^2}}$$

In particular

$$(7) \quad \eta^0 = \frac{u^0}{\sqrt{1 - u^2/c^2}}$$

$$(8) \quad = \frac{c}{\sqrt{1 - u^2/c^2}}$$

Note that the components η^i can be larger than c . This is OK because the four-velocity is the derivative of distance in one coordinate system with respect to the time coordinate in another system, so we're not actually calculating the speed of an object as measured in one specific frame (unless that frame is the object's own rest frame, in which case $u = 0$ and all components η^i are safely less than c).

If we sum up the squares of the spatial components, we get

$$(9) \quad \eta^2 \equiv \sum_{i=1}^3 (\eta^i)^2 = \frac{u^2}{1 - u^2/c^2}$$

$$(10) \quad u^2 = \frac{\eta^2}{1 + \eta^2/c^2}$$

$$(11) \quad \mathbf{u} = \frac{\boldsymbol{\eta}}{\sqrt{1 + \eta^2/c^2}}$$

where bold-face symbols represent 3-d (spatial) vectors.

We can express η in terms of rapidity θ , defined as

$$(12) \quad \frac{u}{c} \equiv \tanh \theta$$

We get, assuming the motion is in the x direction (using $\cosh^2 \theta - \sinh^2 \theta = 1$ in the denominator):

$$(13) \quad \eta^2 = \frac{c^2 \tanh^2 \theta}{1 - \tanh^2 \theta}$$

$$(14) \quad = c^2 \sinh^2 \theta$$

$$(15) \quad \eta = c \sinh \theta$$

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