FOUR-VELOCITY VERSUS ORDINARY VELOCITY: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 25.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c=1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.

Here's an example of calculating the ordinary and four-velocities of an object. Suppose the object moves with ordinary velocity (as measured in the ground frame) $\frac{2}{\sqrt{5}}c$ in the 45° direction (towards the upper right; note that there is a typo in Griffiths's statement of the problem as he states that the velocity is $2\sqrt{5}c$ which is, of course, greater than c). This resolves into components:

$$\mathbf{u}=\sqrt{\frac{2}{5}}c\hat{\mathbf{x}}+\sqrt{\frac{2}{5}}c\hat{\mathbf{y}} \tag{1}$$

The four velocity is defined as

$$\eta^{i}=\frac{u^{i}}{\sqrt{1-u^{2}/c^{2}}} \tag{2}$$

so the components are

$$\eta_{x}=\eta_{y}=\sqrt{2}c \tag{3}$$

We now introduce another frame moving in the x direction with ordinary speed $\sqrt{\frac{2}{5}}c$. The ordinary velocity of the object transforms using the velocity addition formulas

$$\dot{\bar{x}} = \frac{\dot{x}-v}{1-\dot{x}v/c^2} \tag{4}$$

$$\dot{\bar{y}} = \frac{\dot{y}}{\gamma(1-\dot{x}v/c^2)} \tag{5}$$

so we get, using $\gamma = 1/\sqrt{1-\frac{2}{5}} = \sqrt{\frac{5}{3}}$:

$$\bar{u}_x = 0 \tag{6}$$

$$\bar{u}_y = \frac{\sqrt{2/5}}{\sqrt{5/3}\,(1-2/5)}c \tag{7}$$

$$= \sqrt{\frac{2}{3}}c \tag{8}$$

As η^i is a four-vector, it transforms using Lorentz transformations, for which we need

$$\eta^0 = \frac{c}{\sqrt{1-u^2/c^2}} = \sqrt{5}c \tag{9}$$

so

$$\bar{\eta}_x = \gamma\left(\eta_x - \beta\eta^0\right) \tag{10}$$

$$= \sqrt{\frac{5}{3}}\left(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{5}c\right) = 0 \tag{11}$$

$$\bar{\eta}_y = \eta_y = \sqrt{2}c \tag{12}$$

We can check this using 2:

$$\bar{\eta}_x = \frac{\bar{u}_x}{\sqrt{1-\bar{u}^2/c^2}} \tag{13}$$

$$= 0 \tag{14}$$

$$\bar{\eta}_y = \frac{\bar{u}_y}{\sqrt{1-\bar{u}^2/c^2}} \tag{15}$$

$$= \frac{\sqrt{2/3}c}{\sqrt{1/3}} \tag{16}$$

$$= \sqrt{2}c \tag{17}$$

so it checks out.