FOUR-VELOCITY VERSUS ORDINARY VELOCITY: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 25.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.

Here's an example of calculating the ordinary and four-velocities of an object. Suppose the object moves with ordinary velocity (as measured in the ground frame) $\frac{2}{\sqrt{5}}c$ in the 45° direction (towards the upper right; note that there is a typo in Griffiths's statement of the problem as he states that the velocity is $2\sqrt{5}c$ which is, of course, greater than c). This resolves into components:

$$\mathbf{u} = \sqrt{\frac{2}{5}}c\hat{\mathbf{x}} + \sqrt{\frac{2}{5}}c\hat{\mathbf{y}} \tag{1}$$

The four velocity is defined as

$$\eta^i = \frac{u^i}{\sqrt{1 - u^2/c^2}} \tag{2}$$

so the components are

$$\eta_x = \eta_y = \sqrt{2}c \tag{3}$$

We now introduce another frame moving in the x direction with ordinary speed $\sqrt{\frac{2}{5}}c$. The ordinary velocity of the object transforms using the velocity addition formulas

$$\dot{\bar{x}} = \frac{\dot{x} - v}{1 - \dot{x}v/c^2} \tag{4}$$

$$\dot{\bar{y}} = \frac{\dot{y}}{\gamma(1 - \dot{x}v/c^2)} \tag{5}$$

so we get, using $\gamma = 1/\sqrt{1-\frac{2}{5}} = \sqrt{\frac{5}{3}}$:

$$\begin{aligned}
\bar{u}_x &= 0 && (6)\\
\bar{u}_y &= \frac{\sqrt{2/5}}{\sqrt{5/3}\,(1-2/5)}c && (7)\\
&= \sqrt{\frac{2}{3}}c && (8)
\end{aligned}$$

As η^i is a four-vector, it transforms using Lorentz transformations, for which we need

$$\eta^0 = \frac{c}{\sqrt{1-u^2/c^2}} = \sqrt{5}c \tag{9}$$

so

$$\begin{aligned}
\bar{\eta}_x &= \gamma\left(\eta_x - \beta\eta^0\right) && (10)\\
&= \sqrt{\frac{5}{3}}\left(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{5}c\right) = 0 && (11)\\
\bar{\eta}_y &= \eta_y = \sqrt{2}c && (12)
\end{aligned}$$

We can check this using 2:

$$\begin{aligned}
\bar{\eta}_x &= \frac{\bar{u}_x}{\sqrt{1-\bar{u}^2/c^2}} && (13)\\
&= 0 && (14)\\
\bar{\eta}_y &= \frac{\bar{u}_y}{\sqrt{1-\bar{u}^2/c^2}} && (15)\\
&= \frac{\sqrt{2/3}c}{\sqrt{1/3}} && (16)\\
&= \sqrt{2}c && (17)
\end{aligned}$$

so it checks out.