

FOUR-VELOCITY VERSUS ORDINARY VELOCITY: AN EXAMPLE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 25.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.

Here's an example of calculating the ordinary and four-velocities of an object. Suppose the object moves with ordinary velocity (as measured in the ground frame) $\frac{2}{\sqrt{5}}c$ in the 45° direction (towards the upper right; note that there is a typo in Griffiths's statement of the problem as he states that the velocity is $2\sqrt{5}c$ which is, of course, greater than c). This resolves into components:

$$(0.1) \quad \mathbf{u} = \sqrt{\frac{2}{5}}c\hat{\mathbf{x}} + \sqrt{\frac{2}{5}}c\hat{\mathbf{y}}$$

The four velocity is defined as

$$(0.2) \quad \eta^i = \frac{u^i}{\sqrt{1 - u^2/c^2}}$$

so the components are

$$(0.3) \quad \eta_x = \eta_y = \sqrt{2}c$$

We now introduce another frame moving in the x direction with ordinary speed $\sqrt{\frac{2}{5}}c$. The ordinary velocity of the object transforms using the velocity addition formulas

$$(0.4) \quad \dot{x} = \frac{\dot{x} - v}{1 - \dot{x}v/c^2}$$

$$(0.5) \quad \dot{y} = \frac{\dot{y}}{\gamma(1 - \dot{x}v/c^2)}$$

so we get, using $\gamma = 1/\sqrt{1 - \frac{2}{5}} = \sqrt{\frac{5}{3}}$:

$$(0.6) \quad \bar{u}_x = 0$$

$$(0.7) \quad \bar{u}_y = \frac{\sqrt{2/5}}{\sqrt{5/3}(1 - 2/5)}c$$

$$(0.8) \quad = \sqrt{\frac{2}{3}}c$$

As η^i is a four-vector, it transforms using Lorentz transformations, for which we need

$$(0.9) \quad \eta^0 = \frac{c}{\sqrt{1 - u^2/c^2}} = \sqrt{5}c$$

so

$$(0.10) \quad \bar{\eta}_x = \gamma(\eta_x - \beta\eta^0)$$

$$(0.11) \quad = \sqrt{\frac{5}{3}} \left(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{5}c \right) = 0$$

$$(0.12) \quad \bar{\eta}_y = \eta_y = \sqrt{2}c$$

We can check this using 0.2:

$$(0.13) \quad \bar{\eta}_x = \frac{\bar{u}_x}{\sqrt{1 - \bar{u}^2/c^2}}$$

$$(0.14) \quad = 0$$

$$(0.15) \quad \bar{\eta}_y = \frac{\bar{u}_y}{\sqrt{1 - \bar{u}^2/c^2}}$$

$$(0.16) \quad = \frac{\sqrt{2/3}c}{\sqrt{1/3}}$$

$$(0.17) \quad = \sqrt{2}c$$

so it checks out.