FOUR-VELOCITY OF A PARTICLE IN HYPERBOLIC MOTION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 27.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c=1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

We can now return to a particle travelling on a hyperbolic trajectory, so its position (one dimensional, on the x axis) is

$$x(t)=\sqrt{b^{2}+c^{2}t^{2}} \tag{1}$$

Here, x and t are the position and time as measured by an observer at rest (so they are not proper time for the particle). We can find the particle's proper time as a function of t by using the relation between time intervals:

$$d\tau=\sqrt{1-u^{2}/c^{2}}dt \tag{2}$$

We have

$$\begin{aligned}
u &= \dot{x} && (3)\\
&= \frac{c^{2}t}{\sqrt{b^{2}+c^{2}t^{2}}} && (4)\\
d\tau &= \sqrt{1-\frac{c^{2}t^{2}}{b^{2}+c^{2}t^{2}}}dt && (5)\\
&= \frac{b}{\sqrt{b^{2}+c^{2}t^{2}}}dt && (6)
\end{aligned}$$

Integrating both sides, we get, taking $\tau=0$ when $t=0$ (the integral can be done by software or looked up as it is a standard integral):

$$\tau = b\int_0^t \frac{1}{\sqrt{b^2+c^2(t')^2}}dt' \tag{7}$$

$$= \frac{b}{c}\ln\left[\frac{1}{b}\left(ct+\sqrt{b^2+c^2t^2}\right)\right] \tag{8}$$

We can write the position as a function of τ by starting with 1 and using this last result:

$$\tau = \frac{b}{c}\ln\left[\frac{1}{b}(ct+x)\right] \tag{9}$$

$$ct = \sqrt{x^2-b^2} \tag{10}$$

$$\tau = \frac{b}{c}\ln\left[\frac{1}{b}\left(\sqrt{x^2-b^2}+x\right)\right] \tag{11}$$

$$be^{c\tau/b} = \sqrt{x^2-b^2}+x \tag{12}$$

$$\left(be^{c\tau/b}-x\right)^2 = x^2-b^2 \tag{13}$$

$$x = \frac{b}{2}\left(e^{c\tau/b}+e^{-c\tau/b}\right) \tag{14}$$

$$= b\cosh\frac{c\tau}{b} \tag{15}$$

For the ordinary velocity, we have

$$u = \frac{c^2t}{\sqrt{b^2+c^2t^2}} \tag{16}$$

$$= c\frac{\sqrt{x^2-b^2}}{x} \tag{17}$$

$$= cb\frac{\sqrt{\cosh^2\frac{c\tau}{b}-1}}{b\cosh\frac{c\tau}{b}} \tag{18}$$

$$= c\tanh\frac{c\tau}{b}$$

The four velocity is defined as

$$\eta^i = \frac{u^i}{\sqrt{1-u^2/c^2}} \tag{19}$$

so we have (I'm assuming that in part (c) of Griffiths's problem, he meant to ask for η^i in terms of τ, not t, as the latter doesn't give anything particularly informative):

$$\begin{aligned}
\eta^0 &= \frac{c}{\sqrt{1-\tanh^2\frac{c\tau}{b}}} &(20)\\
&= c\cosh\frac{c\tau}{b} &(21)\\
\eta_x &= \frac{u}{\sqrt{1-\tanh^2\frac{c\tau}{b}}} &(22)\\
&= \frac{c\tanh\frac{c\tau}{b}}{\sqrt{1-\tanh^2\frac{c\tau}{b}}} &(23)\\
&= c\sinh\frac{c\tau}{b} &(24)
\end{aligned}$$

As a check, we note that

$$\eta_i\eta^i = -c^2\left(\cosh^2\frac{c\tau}{b} - \sinh^2\frac{c\tau}{b}\right) = -c^2 \qquad (25)$$