

## FOUR-VELOCITY OF A PARTICLE IN HYPERBOLIC MOTION

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 27.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of  $c$  in the equations rather than setting  $c = 1$ ). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

We can now return to a particle travelling on a hyperbolic trajectory, so its position (one dimensional, on the  $x$  axis) is

$$(0.1) \quad x(t) = \sqrt{b^2 + c^2 t^2}$$

Here,  $x$  and  $t$  are the position and time as measured by an observer at rest (so they are not proper time for the particle). We can find the particle's proper time as a function of  $t$  by using the relation between time intervals:

$$(0.2) \quad d\tau = \sqrt{1 - u^2/c^2} dt$$

We have

$$(0.3) \quad u = \dot{x}$$

$$(0.4) \quad = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}}$$

$$(0.5) \quad d\tau = \sqrt{1 - \frac{c^2 t^2}{b^2 + c^2 t^2}} dt$$

$$(0.6) \quad = \frac{b}{\sqrt{b^2 + c^2 t^2}} dt$$

Integrating both sides, we get, taking  $\tau = 0$  when  $t = 0$  (the integral can be done by software or looked up as it is a standard integral):

$$(0.7) \quad \tau = b \int_0^t \frac{1}{\sqrt{b^2 + c^2 (t')^2}} dt'$$

$$(0.8) \quad = \frac{b}{c} \ln \left[ \frac{1}{b} \left( ct + \sqrt{b^2 + c^2 t^2} \right) \right]$$

We can write the position as a function of  $\tau$  by starting with 0.1 and using this last result:

$$(0.9) \quad \tau = \frac{b}{c} \ln \left[ \frac{1}{b} (ct + x) \right]$$

$$(0.10) \quad ct = \sqrt{x^2 - b^2}$$

$$(0.11) \quad \tau = \frac{b}{c} \ln \left[ \frac{1}{b} \left( \sqrt{x^2 - b^2} + x \right) \right]$$

$$(0.12) \quad be^{c\tau/b} = \sqrt{x^2 - b^2} + x$$

$$(0.13) \quad \left( be^{c\tau/b} - x \right)^2 = x^2 - b^2$$

$$(0.14) \quad x = \frac{b}{2} \left( e^{c\tau/b} + e^{-c\tau/b} \right)$$

$$(0.15) \quad = b \cosh \frac{c\tau}{b}$$

For the ordinary velocity, we have

$$(0.16) \quad u = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}}$$

$$(0.17) \quad = c \frac{\sqrt{x^2 - b^2}}{x}$$

$$(0.18) \quad = cb \frac{\sqrt{\cosh^2 \frac{c\tau}{b} - 1}}{b \cosh \frac{c\tau}{b}}$$

$$= c \tanh \frac{c\tau}{b}$$

The four velocity is defined as

$$(0.19) \quad \eta^i = \frac{u^i}{\sqrt{1 - u^2/c^2}}$$

so we have (I'm assuming that in part (c) of Griffiths's problem, he meant to

ask for  $\eta^i$  in terms of  $\tau$ , not  $t$ , as the latter doesn't give anything particularly informative):

$$(0.20) \quad \eta^0 = \frac{c}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}}$$

$$(0.21) \quad = c \cosh \frac{c\tau}{b}$$

$$(0.22) \quad \eta_x = \frac{u}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}}$$

$$(0.23) \quad = \frac{c \tanh \frac{c\tau}{b}}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}}$$

$$(0.24) \quad = c \sinh \frac{c\tau}{b}$$

As a check, we note that

$$(0.25) \quad \eta_i \eta^i = -c^2 \left( \cosh^2 \frac{c\tau}{b} - \sinh^2 \frac{c\tau}{b} \right) = -c^2$$