RELATIVISTIC MOMENTUM AND ENERGY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 28.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c=1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

One way of defining linear momentum of an object of mass m in relativity is to multiply the mass by the velocity, but since we have an 'ordinary' velocity and a four-velocity, we need to choose which one to use. It turns out that if we want momentum to be conserved, we need to use the four-velocity. We can see this as follows.

If we define momentum using ordinary velocity, so that

$$\mathbf{p}=m\mathbf{u} \tag{1}$$

we can rework our collision problem from earlier. In an inertial frame $\mathcal{S}$ we have a collision between a particle A (mass m_A and velocity $\mathbf{u}_A$) and another particle B (mass m_B and velocity $\mathbf{u}_B$). During the collision, some of the mass of A gets transferred to B, so that afterwards we have particles C and D with masses m_C, m_D and velocities $\mathbf{u}_C, \mathbf{u}_D$. If momentum is conserved, then

$$m_A\mathbf{u}_A+m_B\mathbf{u}_B = m_C\mathbf{u}_C+m_D\mathbf{u}_D \tag{2}$$

In inertial frame $\bar{\mathcal{S}}$ which moves with velocity $\mathbf{v}$ relative to $\mathcal{S}$, the velocities of the particles transform using the formulas

$$\begin{aligned}
\dot{\bar{x}} &= \frac{\dot{x}-v}{1-\dot{x}v/c^2} && (3)\\
\dot{\bar{y}} &= \frac{\dot{y}}{\gamma(1-\dot{x}v/c^2)} && (4)\\
\dot{\bar{z}} &= \frac{\dot{z}}{\gamma(1-\dot{x}v/c^2)} && (5)
\end{aligned}$$

The LHS of 2 transforms as

$$\frac{m_A}{1-u_{A_x}v/c^2}\left[\left(u_{A_x}-v\right)\hat{\mathbf{x}}+\frac{u_{A_y}}{\gamma}\hat{\mathbf{y}}+\frac{u_{A_z}}{\gamma}\hat{\mathbf{z}}\right]+\frac{m_B}{1-u_{B_x}v/c^2}\left[\left(u_{B_x}-v\right)\hat{\mathbf{x}}+\frac{u_{B_y}}{\gamma}\hat{\mathbf{y}}+\frac{u_{B_z}}{\gamma}\hat{\mathbf{z}}\right] \tag{6}$$

We can't use 2 to convert this into the transform of the RHS (where A is replaced by C and B by D) because of the different factors of $\frac{1}{1-u_{A_x}v/c^2}$ and $\frac{1}{1-u_{B_x}v/c^2}$ multiplying each term, that is, for the x component for example, we can't factor out a term $m_A u_{A_x}+m_B u_{B_x}$ from this expression to set it equal to $m_C u_{C_x}+m_D u_{D_x}$.

If we use the four-velocity to define momentum, however, things work out properly. We define

$$\mathbf{p}=m\eta \tag{7}$$

(remember that Griffiths uses η for four-velocity). To make this a four-vector, we define

$$p^0=m\eta^0=\frac{mc}{\sqrt{1-u^2/c^2}} \tag{8}$$

Now we start with three-momentum (using the spatial components of the four-momentum) conserved in $\mathcal{S}$:

$$m_A\eta_A+m_B\eta_B=m_C\eta_C+m_D\eta_D \tag{9}$$

To convert to $\bar{\mathcal{S}}$ we can use Lorentz transformations, since the masses are scalars (constants) and the ηs are four-vectors. The LHS becomes

$$\begin{aligned}m_A\left[\gamma\left(\eta_A^1-\beta\eta_A^0\right)\hat{\mathbf{x}}+\eta_A^2\hat{\mathbf{y}}+\eta_A^3\hat{\mathbf{z}}\right]+m_B\left[\gamma\left(\eta_B^1-\beta\eta_B^0\right)\hat{\mathbf{x}}+\eta_B^2\hat{\mathbf{y}}+\eta_B^3\hat{\mathbf{z}}\right]=\\ \gamma\hat{\mathbf{x}}\left[m_A\eta_A^1+m_B\eta_B^1-\beta\left(m_A\eta_A^0+m_B\eta_B^0\right)\right]+\hat{\mathbf{y}}\left(m_A\eta_A^2+m_B\eta_B^2\right)+\hat{\mathbf{z}}\left(m_A\eta_A^3+m_B\eta_B^3\right)\end{aligned} \tag{10}$$

If we now apply 9 to each spatial component separately, we see that this last line is equal to

$$\gamma\hat{\mathbf{x}}\left[m_C\eta_C^1+m_D\eta_D^1-\beta\left(m_A\eta_A^0+m_B\eta_B^0\right)\right]+\hat{\mathbf{y}}\left(m_C\eta_C^2+m_D\eta_D^2\right)+\hat{\mathbf{z}}\left(m_C\eta_C^3+m_D\eta_D^3\right) \tag{11}$$

In order for this last expression to be equal to the Lorentz transform of the RHS of 9, the one remaining term containing A and B terms must equal its corresponding term in the transform of the RHS. That is, we must have

$$m_A\eta_A^0+m_B\eta_B^0=m_C\eta_C^0+m_D\eta_D^0 \tag{12}$$

This is used as the motivation to define the relativistic energy as

$$E = cp^0 = mc\eta^0 = \frac{mc^2}{\sqrt{1-u^2/c^2}} \tag{13}$$

With these definitions, we can see that four-momentum (which is 3-momentum and energy together) are conserved.

PINGBACKS

Pingback: Four-velocity's square is a universal constant
Pingback: Relativistic kinetic energy
Pingback: Decay of a pion into a muon and a neutrino
Pingback: Collision of two identical particles
Pingback: Decay of a pion into two photons
Pingback: Pair annihilation with an electron and a positron
Pingback: Force in relativity
Pingback: Collision of a pion and a proton
Pingback: Relativistic energy revisited