

RELATIVISTIC MOMENTUM AND ENERGY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 28.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

One way of defining linear momentum of an object of mass m in relativity is to multiply the mass by the velocity, but since we have an 'ordinary' velocity and a four-velocity, we need to choose which one to use. It turns out that if we want momentum to be conserved, we need to use the four-velocity. We can see this as follows.

If we define momentum using ordinary velocity, so that

$$(0.1) \quad \mathbf{p} = m\mathbf{u}$$

we can rework our collision problem from earlier. In an inertial frame \mathcal{S} we have a collision between a particle A (mass m_A and velocity \mathbf{u}_A) and another particle B (mass m_B and velocity \mathbf{u}_B). During the collision, some of the mass of A gets transferred to B , so that afterwards we have particles C and D with masses m_C, m_D and velocities $\mathbf{u}_C, \mathbf{u}_D$. If momentum is conserved, then

$$(0.2) \quad m_A\mathbf{u}_A + m_B\mathbf{u}_B = m_C\mathbf{u}_C + m_D\mathbf{u}_D$$

In inertial frame $\tilde{\mathcal{S}}$ which moves with velocity \mathbf{v} relative to \mathcal{S} , the velocities of the particles transform using the formulas

$$(0.3) \quad \dot{\tilde{x}} = \frac{\dot{x} - v}{1 - \dot{x}v/c^2}$$

$$(0.4) \quad \dot{\tilde{y}} = \frac{\dot{y}}{\gamma(1 - \dot{x}v/c^2)}$$

$$(0.5) \quad \dot{\tilde{z}} = \frac{\dot{z}}{\gamma(1 - \dot{x}v/c^2)}$$

The LHS of 0.2 transforms as

$$(0.6) \quad \frac{m_A}{1 - u_{A_x}v/c^2} \left[(u_{A_x} - v) \hat{\mathbf{x}} + \frac{u_{A_y}}{\gamma} \hat{\mathbf{y}} + \frac{u_{A_z}}{\gamma} \hat{\mathbf{z}} \right] + \frac{m_B}{1 - u_{B_x}v/c^2} \left[(u_{B_x} - v) \hat{\mathbf{x}} + \frac{u_{B_y}}{\gamma} \hat{\mathbf{y}} + \frac{u_{B_z}}{\gamma} \hat{\mathbf{z}} \right]$$

We can't use 0.2 to convert this into the transform of the RHS (where A is replaced by C and B by D) because of the different factors of $\frac{1}{1 - u_{A_x}v/c^2}$ and $\frac{1}{1 - u_{B_x}v/c^2}$ multiplying each term, that is, for the x component for example, we can't factor out a term $m_A u_{A_x} + m_B u_{B_x}$ from this expression to set it equal to $m_C u_{C_x} + m_D u_{D_x}$.

If we use the four-velocity to define momentum, however, things work out properly. We define

$$(0.7) \quad \mathbf{p} = m\boldsymbol{\eta}$$

(remember that Griffiths uses $\boldsymbol{\eta}$ for four-velocity). To make this a four-vector, we define

$$(0.8) \quad p^0 = m\eta^0 = \frac{mc}{\sqrt{1 - u^2/c^2}}$$

Now we start with three-momentum (using the spatial components of the four-momentum) conserved in \mathcal{S} :

$$(0.9) \quad m_A \boldsymbol{\eta}_A + m_B \boldsymbol{\eta}_B = m_C \boldsymbol{\eta}_C + m_D \boldsymbol{\eta}_D$$

To convert to $\bar{\mathcal{S}}$ we can use Lorentz transformations, since the masses are scalars (constants) and the $\boldsymbol{\eta}$ s are four-vectors. The LHS becomes

$$(0.10) \quad m_A [\gamma(\eta_A^1 - \beta\eta_A^0) \hat{\mathbf{x}} + \eta_A^2 \hat{\mathbf{y}} + \eta_A^3 \hat{\mathbf{z}}] + m_B [\gamma(\eta_B^1 - \beta\eta_B^0) \hat{\mathbf{x}} + \eta_B^2 \hat{\mathbf{y}} + \eta_B^3 \hat{\mathbf{z}}] = \\ \gamma \hat{\mathbf{x}} [m_A \eta_A^1 + m_B \eta_B^1 - \beta(m_A \eta_A^0 + m_B \eta_B^0)] + \hat{\mathbf{y}} (m_A \eta_A^2 + m_B \eta_B^2) + \hat{\mathbf{z}} (m_A \eta_A^3 + m_B \eta_B^3)$$

If we now apply 0.9 to each spatial component separately, we see that this last line is equal to

$$(0.11) \quad \gamma \hat{\mathbf{x}} [m_C \eta_C^1 + m_D \eta_D^1 - \beta(m_A \eta_A^0 + m_B \eta_B^0)] + \hat{\mathbf{y}} (m_C \eta_C^2 + m_D \eta_D^2) + \hat{\mathbf{z}} (m_C \eta_C^3 + m_D \eta_D^3)$$

In order for this last expression to be equal to the Lorentz transform of the RHS of 0.9, the one remaining term containing A and B terms must equal its corresponding term in the transform of the RHS. That is, we must have

$$(0.12) \quad m_A \eta_A^0 + m_B \eta_B^0 = m_C \eta_C^0 + m_D \eta_D^0$$

This is used as the motivation to define the relativistic energy as

$$(0.13) \quad E = cp^0 = mc\eta^0 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

With these definitions, we can see that four-momentum (which is 3-momentum and energy together) are conserved.

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