RELATIVISTIC KINETIC ENERGY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 29.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c=1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

We can use the definition of relativistic energy to further define kinetic energy. The relativistic total energy is

$$E=\frac{mc^{2}}{\sqrt{1-u^{2}/c^{2}}} \tag{0.1}$$

where m is the rest mass and u is the object's velocity in some inertial frame. If $u=0$, then we get the rest energy (the most famous formula in physics):

$$E=mc^{2} \tag{0.2}$$

Since all the rest of the energy is due to the object's motion, it is natural to define this excess energy as the kinetic energy:

$$K\equiv E-mc^{2}=mc^{2}\left(\frac{1}{\sqrt{1-u^{2}/c^{2}}}-1\right)=mc^{2}\left(\frac{\eta^{0}}{c}-1\right) \tag{0.3}$$

where η is the four-velocity.

Using this definition, we can show that if a collision is elastic in one frame $\mathscr{S}$ it is also elastic in another frame $\bar{\mathscr{S}}$. Using the same example we treated earlier, since K is conserved in $\mathscr{S}$, we have

$$m_{A}c^{2}\left(\frac{\eta_{A}^{0}}{c}-1\right)+m_{B}c^{2}\left(\frac{\eta_{B}^{0}}{c}-1\right)=m_{C}c^{2}\left(\frac{\eta_{C}^{0}}{c}-1\right)+m_{D}c^{2}\left(\frac{\eta_{D}^{0}}{c}-1\right) \tag{0.4}$$

Since total energy is conserved, we have

$$m_A\eta_A^0 + m_B\eta_B^0 = m_C\eta_C^0 + m_D\eta_D^0 \tag{0.5}$$

so cancelling these terms in 0.4, we find that rest mass is also conserved:

$$m_A + m_B = m_C + m_D \tag{0.6}$$

Using a Lorentz transformation on the LHS of 0.4, we have in frame $\bar{\mathscr{S}}$:

$$m_A\left(\frac{\gamma}{c}\left(\eta_A^0 - \beta\eta_A^1\right) - 1\right) + m_B\left(\frac{\gamma}{c}\left(\eta_B^0 - \beta\eta_B^1\right) - 1\right) \tag{0.7}$$

Using conservation of energy 0.5 and rest mass 0.6, along with conservation of the x component of momentum, that is:

$$m_A\eta_A^1 + m_B\eta_B^1 = m_C\eta_C^1 + m_D\eta_D^1 \tag{0.8}$$

we get

$$m_A\left(\frac{\gamma}{c}\left(\eta_A^0 - \beta\eta_A^1\right) - 1\right) + m_B\left(\frac{\gamma}{c}\left(\eta_B^0 - \beta\eta_B^1\right) - 1\right) = \ldots \tag{0.9}$$

$$\frac{\gamma}{c}\left(m_A\eta_A^0 + m_B\eta_B^0\right) - \frac{\beta\gamma}{c}\left(m_A\eta_A^1 + m_B\eta_B^1\right) - (m_A + m_B) = \ldots \tag{0.10}$$

$$\frac{\gamma}{c}\left(m_C\eta_C^0 + m_D\eta_D^0\right) - \frac{\beta\gamma}{c}\left(m_C\eta_C^1 + m_D\eta_D^1\right) - (m_C + m_D) \tag{0.11}$$

Thus the collision is also elastic in $\bar{\mathscr{S}}$.

Example. If a particle's kinetic energy is n times its rest energy, then from 0.3

$$\frac{1}{\sqrt{1-u^2/c^2}} - 1 = n \tag{0.12}$$

$$u = c\sqrt{1-\frac{1}{(n+1)^2}} \tag{0.13}$$

Thus if $n = 0$ (no kinetic energy), then $u = 0$ (the particle is at rest), and as $n \to \infty$, $u \to c$ so the larger the kinetic energy, the closer the particle's speed gets to c.