

## RELATIVISTIC KINETIC ENERGY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 29.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of  $c$  in the equations rather than setting  $c = 1$ ). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

We can use the definition of relativistic energy to further define kinetic energy. The relativistic total energy is

$$(1) \quad E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

where  $m$  is the rest mass and  $u$  is the object's velocity in some inertial frame. If  $u = 0$ , then we get the rest energy (the most famous formula in physics):

$$(2) \quad E = mc^2$$

Since all the rest of the energy is due to the object's motion, it is natural to define this excess energy as the kinetic energy:

$$(3) \quad K \equiv E - mc^2 = mc^2 \left( \frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right) = mc^2 \left( \frac{\eta^0}{c} - 1 \right)$$

where  $\eta$  is the four-velocity.

Using this definition, we can show that if a collision is elastic in one frame  $\mathcal{S}$  it is also elastic in another frame  $\mathcal{S}'$ . Using the same example we treated earlier, since  $K$  is conserved in  $\mathcal{S}$ , we have

$$(4) \quad m_A c^2 \left( \frac{\eta_A^0}{c} - 1 \right) + m_B c^2 \left( \frac{\eta_B^0}{c} - 1 \right) = m_C c^2 \left( \frac{\eta_C^0}{c} - 1 \right) + m_D c^2 \left( \frac{\eta_D^0}{c} - 1 \right)$$

Since total energy is conserved, we have

$$(5) \quad m_A \eta_A^0 + m_B \eta_B^0 = m_C \eta_C^0 + m_D \eta_D^0$$

so cancelling these terms in 4, we find that rest mass is also conserved:

$$(6) \quad m_A + m_B = m_C + m_D$$

Using a Lorentz transformation on the LHS of 4, we have in frame  $\bar{\mathcal{S}}$ :

$$(7) \quad m_A \left( \frac{\gamma}{c} (\eta_A^0 - \beta \eta_A^1) - 1 \right) + m_B \left( \frac{\gamma}{c} (\eta_B^0 - \beta \eta_B^1) - 1 \right)$$

Using conservation of energy 5 and rest mass 6, along with conservation of the  $x$  component of momentum, that is:

$$(8) \quad m_A \eta_A^1 + m_B \eta_B^1 = m_C \eta_C^1 + m_D \eta_D^1$$

we get

$$(9) \quad m_A \left( \frac{\gamma}{c} (\eta_A^0 - \beta \eta_A^1) - 1 \right) + m_B \left( \frac{\gamma}{c} (\eta_B^0 - \beta \eta_B^1) - 1 \right) = \dots$$

$$(10) \quad \frac{\gamma}{c} (m_A \eta_A^0 + m_B \eta_B^0) - \frac{\beta \gamma}{c} (m_A \eta_A^1 + m_B \eta_B^1) - (m_A + m_B) = \dots$$

$$(11) \quad \frac{\gamma}{c} (m_C \eta_C^0 + m_D \eta_D^0) - \frac{\beta \gamma}{c} (m_C \eta_C^1 + m_D \eta_D^1) - (m_C + m_D)$$

Thus the collision is also elastic in  $\bar{\mathcal{S}}$ .

**Example.** If a particle's kinetic energy is  $n$  times its rest energy, then from 3

$$(12) \quad \frac{1}{\sqrt{1 - u^2/c^2}} - 1 = n$$

$$(13) \quad u = c \sqrt{1 - \frac{1}{(n+1)^2}}$$

Thus if  $n = 0$  (no kinetic energy), then  $u = 0$  (the particle is at rest), and as  $n \rightarrow \infty$ ,  $u \rightarrow c$  so the larger the kinetic energy, the closer the particle's speed gets to  $c$ .