

RELATIVISTIC KINETIC ENERGY

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Post 29.

[Griffiths's approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of c in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I'll use his notation here, but anyone attempting to follow both books should beware.]

We can use the definition of relativistic energy to further define kinetic energy. The relativistic total energy is

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \quad (1)$$

where m is the rest mass and u is the object's velocity in some inertial frame. If $u = 0$, then we get the rest energy (the most famous formula in physics):

$$E = mc^2 \quad (2)$$

Since all the rest of the energy is due to the object's motion, it is natural to define this excess energy as the kinetic energy:

$$K \equiv E - mc^2 = mc^2 \left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right) = mc^2 \left(\frac{\eta^0}{c} - 1 \right) \quad (3)$$

where η is the four-velocity.

Using this definition, we can show that if a collision is elastic in one frame \mathcal{S} it is also elastic in another frame $\bar{\mathcal{S}}$. Using the same example we treated earlier, since K is conserved in \mathcal{S} , we have

$$m_A c^2 \left(\frac{\eta_A^0}{c} - 1 \right) + m_B c^2 \left(\frac{\eta_B^0}{c} - 1 \right) = m_C c^2 \left(\frac{\eta_C^0}{c} - 1 \right) + m_D c^2 \left(\frac{\eta_D^0}{c} - 1 \right) \quad (4)$$

Since total energy is conserved, we have

$$m_A \eta_A^0 + m_B \eta_B^0 = m_C \eta_C^0 + m_D \eta_D^0 \quad (5)$$

so cancelling these terms in 4, we find that rest mass is also conserved:

$$m_A + m_B = m_C + m_D \quad (6)$$

Using a Lorentz transformation on the LHS of 4, we have in frame \bar{S} :

$$m_A \left(\frac{\gamma}{c} (\eta_A^0 - \beta \eta_A^1) - 1 \right) + m_B \left(\frac{\gamma}{c} (\eta_B^0 - \beta \eta_B^1) - 1 \right) \quad (7)$$

Using conservation of energy 5 and rest mass 6, along with conservation of the x component of momentum, that is:

$$m_A \eta_A^1 + m_B \eta_B^1 = m_C \eta_C^1 + m_D \eta_D^1 \quad (8)$$

we get

$$m_A \left(\frac{\gamma}{c} (\eta_A^0 - \beta \eta_A^1) - 1 \right) + m_B \left(\frac{\gamma}{c} (\eta_B^0 - \beta \eta_B^1) - 1 \right) = \dots \quad (9)$$

$$\frac{\gamma}{c} (m_A \eta_A^0 + m_B \eta_B^0) - \frac{\beta \gamma}{c} (m_A \eta_A^1 + m_B \eta_B^1) - (m_A + m_B) = \dots \quad (10)$$

$$\frac{\gamma}{c} (m_C \eta_C^0 + m_D \eta_D^0) - \frac{\beta \gamma}{c} (m_C \eta_C^1 + m_D \eta_D^1) - (m_C + m_D) \quad (11)$$

Thus the collision is also elastic in \bar{S} .

Example. If a particle's kinetic energy is n times its rest energy, then from 3

$$\frac{1}{\sqrt{1 - u^2/c^2}} - 1 = n \quad (12)$$

$$u = c \sqrt{1 - \frac{1}{(n+1)^2}} \quad (13)$$

Thus if $n = 0$ (no kinetic energy), then $u = 0$ (the particle is at rest), and as $n \rightarrow \infty$, $u \rightarrow c$ so the larger the kinetic energy, the closer the particle's speed gets to c .